ON THE APSIDAL LIMITS OF A ROLLING MISSILE*

BY

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Abstract. It is proved that a rolling missile whose initial angular oscillations are nonlinear will have the same librations as its equivalent common top provided that $q > 0$ and $Z_1 + Z_4 \leq -2$ where q is certain aerodynamic parameter contained in the nonlinear overturning moment of the missile and Z_1, Z_4 are respectively the least negative and the largest positive zeros of a certain quartic polynomial.

Introduction. Angular oscillations of a rolling missile round its center of mass are usually simulated by the oscillations of a common top which is dynamically similar to the missile. Small vibrations of the missile are known to have qualitative and quantitative agreements with those of an equivalent common top. However, due to specific launching conditions or inadequacy of spin, whenever the amplitudes of initial oscillations of the missile become large, this analogy breaks down (see Rath and Namboodiri [8, 9]). Under such circumstances, the reliability of the simulation may be questioned so long as one has not established conditions characterizing the validity of the analogy of the two motions.

The total precessional advance of the missile, as its axis moves from one stationary state to another, has certain bounds, like the Kohn-Hadamard [7, 5] bounds of apses of a common top. In case of nonlinear angular motion, the principal aerodynamic moment of the missile depends strongly on the angle of nutation and as such the apsidal limits of the missile are not the same as that of an equivalent common top. Even precession and spin do not always have the same sign, as will be proved in Sec. 8.

Having made these observations, we have laid down conditions under which the apsidal limits of the missile should exist and be the same as that of the common top. The apsidal limits of the common top are independent of initial conditions, whereas for the missile certain launching conditions have to be restricted. This is expressed by the fact that the sum of the smallest (negative) and the largest (positive) roots of the libration polynomial (2.10) should have a negative upper bound. More precisely, we have proved the following:

THEOREM. A missile will have the same apsidal limits as the equivalent common top if, with reference to its libration polynomial (2.10), we stipulate that

$$q > 0, \quad z_1 + z_4 \leq -2$$

where z_1 and z_4 are the smallest and the largest roots of the polynomial. This result is conveyed through Sec. 7 and the various theorems are proved in Sec. 6. Our entire analysis is based on a simple geometrical method outlined by Diaz and Metcalf [2, 3]. We do not claim that the sufficient conditions laid down in the various theorems are always the best possible ones.

2. Equations of motion. Consider the angular oscillations of a missile about its center of mass. The differential equations governing such motions are given by [4]

$$\delta'^2 + \phi'^2 \sin^2 \delta - (2/B) \int_0^\delta M \, d\delta = E \tag{2.1}$$

and

$$\phi' \sin^2 \delta + \Omega \cos \delta = F \tag{2.2}$$

where M is the aerodynamic upsetting moment given by

$$M = \mu(\delta) \sin \delta \tag{2.3}$$

with

$$\mu(\delta) = (B\Omega^2/4s)(1 - 4qs + 4qs \cos \delta) \tag{2.4}$$

and E and F are constants of integration, δ is the angle of nutation and ϕ the angle of precession. Ω and s are given by

$$\Omega = (AN/B) \quad \text{and} \quad s = (A^2N^2/4B\mu(0)) \tag{2.5}$$

where A and B are the axial and transverse moments of inertia of the projectile which has an axis of dynamic and aerodynamic symmetry and N is the constant axial spin. Here q is a dimensionless parameter assumed positive for the present analysis. A prime signifies time derivative.

The elimination of ϕ' between Eqs. (2.1) and (2.2) leads to

$$\delta'^2 \sin^2 \delta + (F - \Omega \cos \delta)^2 + (\Omega^2 \sin^2 \delta/2s) \{(1 - 4qs) \cos \delta +$$
$$+ 2qs \cos^2 \delta\} = E, \tag{2.6}$$

the well-known Lock-Fowler equations of the yawing motion of the missile.

Now, setting

$$z = (1 + \cos \delta)/2, \tag{2.7}$$

Eqs. (2.6) and (2.2) become

$$z'^2 = H(z) \tag{2.8}$$

and

$$\phi' = \Omega(\lambda - z)/2z(1 - z) \tag{2.9}$$

where

$$H(z) = z(1 - z)\{E - \alpha(2z - 1) - \beta(2z - 1)^2\} - \Omega^2(\lambda - z)^2 \tag{2.10}$$

and

$$\lambda = (F + \Omega)/2\Omega. \tag{2.11}$$

The constants α and β appearing in Eq. (2.10) are given by

$$\alpha = (\Omega^2/2s)(1 - 4qs) \quad \text{and} \quad \beta = q\Omega^2 \tag{2.12}$$

It may be noted that for real motion the zeros z_i ($i = 1$ to 4) of $H(z)$ are real and satisfy the following inequalities:

$$z_1 \leq 0 \leq z_2 < z < z_3 \leq 1 \leq z_4. \tag{2.13}$$

With this (2.8) becomes

$$z'^2 = 4\beta(z - z_1)(z - z_2)(z_3 - z)(z_4 - z), \tag{2.14}$$

whence

$$z_1 + z_2 + z_3 + z_4 = 2 - (\alpha/2\beta), \tag{2.15}$$

$$z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_3z_4 + z_2z_4 = (5\beta - E - 3\alpha - \Omega^2)\,4\beta, \tag{2.16}$$

$$z_1z_2z_3 + z_1z_2z_4 + z_2z_3z_4 + z_1z_3z_4 = (\beta - E - \alpha - 2\Omega^2\lambda)/4\beta \tag{2.17}$$

and

$$z_1z_2z_3z_4 = -\Omega^2\lambda^2/4\beta, \tag{2.18}$$

By subtracting (2.17) from (2.16), it follows that

$$z_1(z_2 + z_3 + z_4) + (z_2z_3 + z_3z_4 + z_2z_4)(1 - z) - z_2z_3z_4 = 1 - (\alpha/2\beta) + \Omega^2(2\lambda - 1)/4\beta.$$

Using (2.15) and (2.18) again, we obtain

$$(z_1 - 1)\,\{(z_2 + z_3z_4 - 1) - (z_2z_3 + z_3z_4 + z_2z_4)\} = z_2z_3z_4\,\{1 - z_1(2\lambda - 1)/\lambda^2\} \tag{2.19}$$

from which we get

$$z_1 - 1 = z_2z_3z_4\,\{1 - (2\lambda - 1)/\lambda^2\}/D_1 \tag{2.20}$$

and

$$z_1 = D_2/D_1 \tag{2.21}$$

with

$$D_1 = (z_2 + z_3 + z_4 - 1) - (z_2z_3 + z_3z_4 + z_2z_4) + z_2z_3z_4(2\lambda - 1)/\lambda^2$$

and

$$D_2 = (z_2 + z_3 + z_4 - 1) - (z_2z_3 + z_3z_4 + z_2z_4) + z_2z_3z_4,$$

so that

$$\frac{z_1 - 1}{z_1} = \left(\frac{\lambda - 1}{\lambda}\right)^2 \left(\frac{z_2z_3z_4}{D_2}\right).$$

But

$$D_2 = (1 - z_2)(1 - z_3)(z_4 - 1)$$

and hence we have finally

$$|\lambda|\,\{(1 - z_1)(1 - z_2)(1 - z_3)(z_4 - 1)\}^{1/2} = |\lambda - 1|\,(-z_1z_2z_3z_4)^{1/2}. \tag{2.22}$$

Also, (2.19) can be rewritten as

$$z_2 + z_3 = 1 + \frac{z_2 z_3 \{1 + z_1 z_4 (2\lambda - 1)/\lambda^2 - (z_1 + z_4)\}}{(1 - z_4)(1 - z_1)} \tag{2.23}$$

Hence, whenever

$$z_1 + z_4 < 1 + z_1 z_4 (2\lambda - 1)/\lambda^2, \tag{2.24}$$

$$z_2 + z_3 < 1, \tag{2.25}$$

a property which is generally observed in the case of an ordinary gyroscope for $\lambda \leq 1/2$ (cf. [3, Eq. (1.14)]). It may be noted that inequality (2.24) is always satisfied whenever $z_1 + z_4 \leq 0$ and $\lambda \leq 1/2$.

3. The apsidal angle.

From (2.8) and (2.9) the apsidal angle of a Lock-Fowler missile is given by

$$\Phi = \int_{z_2}^{z_3} \frac{\Omega(\lambda - z)\, dz}{2z(1 - z)(H(z))^{1/2}}$$

$$= \frac{sgn\ \Omega}{2\,|\lambda|} \int_{z_2}^{z_3} \frac{(-z_1 z_2 z_3 z_4)^{1/2} h(z)\, dz}{z(1 - z)\{(z - z_2)(z_3 - z)\}^{1/2}} \tag{3.1}$$

due to (2.18). In (3.1)

$$h(z) = (\lambda - z)/\{(z - z_1)(z_4 - z)\}^{1/2}. \tag{3.2}$$

Splitting $(\lambda - z)/z(1 - z)$ into partial fractions and using (2.22), we have, for $\Omega > 0$

$$\Phi = sgn\ (\lambda)\Phi_1 + sgn\ (\lambda - 1)\ \Phi_2 \tag{3.3}$$

where

$$\Phi_1 = (1/2) \int_{z_2}^{z_3} \frac{(-z_1 z_2 z_3 z_4)^{1/2}\ l(z)\ dz}{z\{(z - z_2)(z_3 - z)\}^{1/2}} \tag{3.4}$$

and

$$\Phi_2 = (1/2) \int_{z_2}^{z_3} \frac{\{(1 - z_1)(1 - z_2)(1 - z_3)(z_4 - 1)\}^{1/2}\ l(z)\ dz}{(1 - z)\{(z - z_2)(z_3 - z)\}^{1/2}} \tag{3.5}$$

with

$$l(z) = \{(z - z_1)(z_4 - z)\}^{-1/2}. \tag{3.6}$$

4. Certain geometrical propositions on the functions $h(z)$ and $l(z)$.

In order to establish the bounds for Φ we need to prove certain elementary geometrical propositions regarding the functions $h(z)$, and $l(z)$ defined in (3.2) and (3.6). It may be noted that the graphs of these functions have the asymptotes $z = z_i$ $(i = 1, 4)$ and are continuous over the interval $z_1 < z < z_4$.

At any point of the curve (3.2) we have

$$\{(z - z_1)(z_4 - z)\}^{3/2}, \quad dh(z)/dz = az + b \tag{4.1}$$

and

$$\{(z - z_1)(z_4 - z)\}^{5/2}, \quad d^2h(z)/dz^2 = 2az^2 + cz + d \tag{4.2}$$

where

$$a = 2\lambda - (z_1 + z_4), \tag{4.3}$$

$$b = 2z_1z_4 - \lambda(z_1 + z_4), \tag{4.4}$$

$$2c = (z_1 + z_4)^2 - 8\lambda(z_1 + z_4) + 12z_1z_4, \tag{4.5}$$

$$2d = 3\lambda(z_1 + z_4)^2 - 4z_1z_4(\lambda + z_1 + z_4). \tag{4.6}$$

Therefore, if we denote the stationary point of (3.2) by z^* and the points of inflection by z_i^{**} ($i = 1, 2$), they are given by

$$z^* = -(b/a) = \{\lambda(z_1 + z_4) - 2z_1z_4\}/\{2\lambda - (z_1 + z_4)\} \tag{4.7}$$

and

$$z_i^{**} = (-c \pm (c^2 - 8ad)^{1/2})/4a = [\{8\lambda(z_1 + z_4) + (z_1 + z_4)^2 - 12z_1z_4\}$$
$$\pm (z_4 - z_1)\{32(\lambda - \lambda_1)(\lambda_2 - \lambda)\}^{1/2}]/8\{2\lambda - (z_1 + z_4)\} \tag{4.8}$$

where

$$\lambda_1 = \tfrac{1}{2}\{(z_1 + z_4) + 3\sqrt{2}(z_1 - z_4)/4\}, \tag{4.9}$$

$$\lambda_2 = \tfrac{1}{2}\{(z_1 + z_4) - 3\sqrt{2}(z_1 - z_4)/4\}. \tag{4.10}$$

Accordingly, we shall write (4.1) and (4.2) as

$$\{(z - z_1)(z_4 - z)\}^{3/2} \cdot dh(z)/dz = a(z - z^*) \tag{4.11}$$

and

$$\{(z - z_1)(z_4 - z)\}^{5/2} \cdot d^2h(z)/d^2z = 2a(z - z_1^{**})(z - z_2^{**}). \tag{4.12}$$

Further, from (3.2) and (4.7) we also have

$$h(z^*) = 2(\lambda - z_1)(\lambda - z_4)/a\{(z^* - z_1)(z_4 - z^*)\}^{1/2}. \tag{4.13}$$

Evidently the points of inflexion are real iff $\lambda_1 \leq \lambda \leq \lambda_2$, and in the particular cases when $\lambda = \lambda_i$ ($i = 1, 2$) $z_1^{**} = z_2^{**}$ and it corresponds to a point of undulation on the curve (3.2) [10].

It may be noted that

$$h(z) = -2\{(z - z_1)/(z_4 - z)\}^{1/2}, \quad \text{when} \quad \lambda = z_1$$
$$= 2\{(z_4 - z)/(z - z_1)\}^{1/2}, \quad \text{when} \quad \lambda = z_4$$
$$= -\{(z - z_1)/(z_4 - z)\}^{1/2} + \{(z_4 - z)/(z - z_1)\}^{1/2}, \quad \text{when} \quad a = 0. \tag{4.14}$$

The general form of the curve of $h(z)$ for different cases (discussed in the present communication) is shown schematically in the following diagrams.
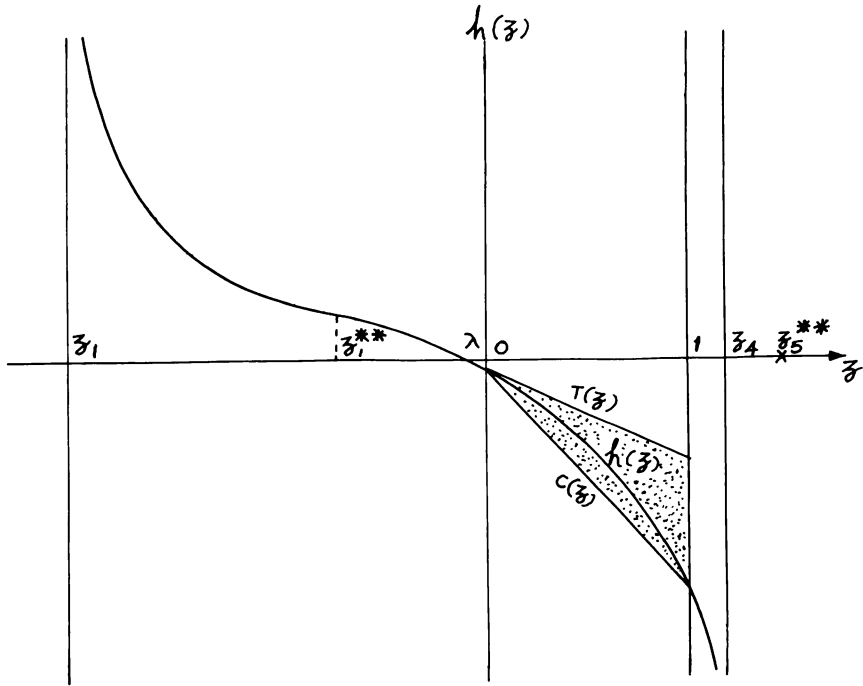
FIG. 1. $[a > 0; z_1 < \lambda < \frac{2}{3} < z_4]$.
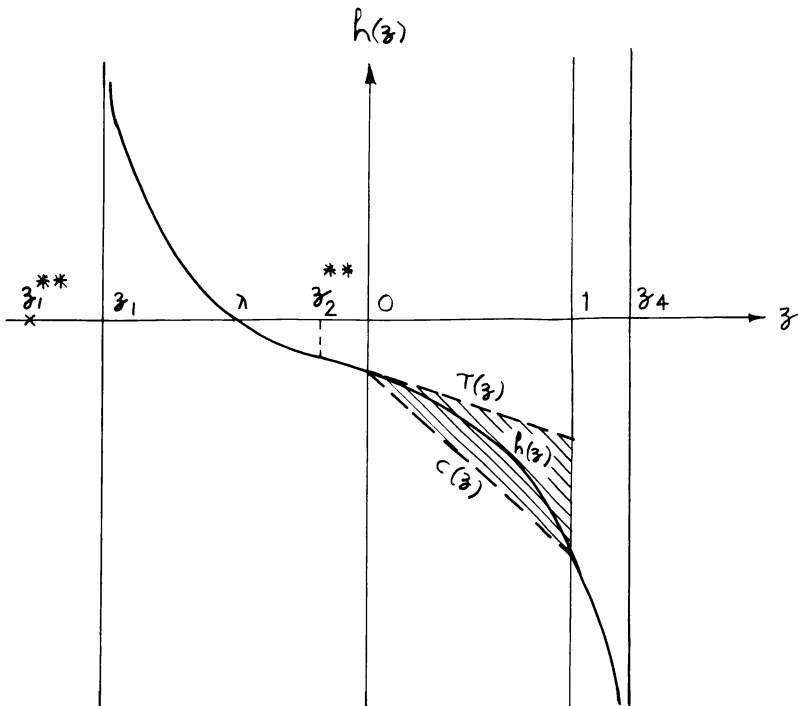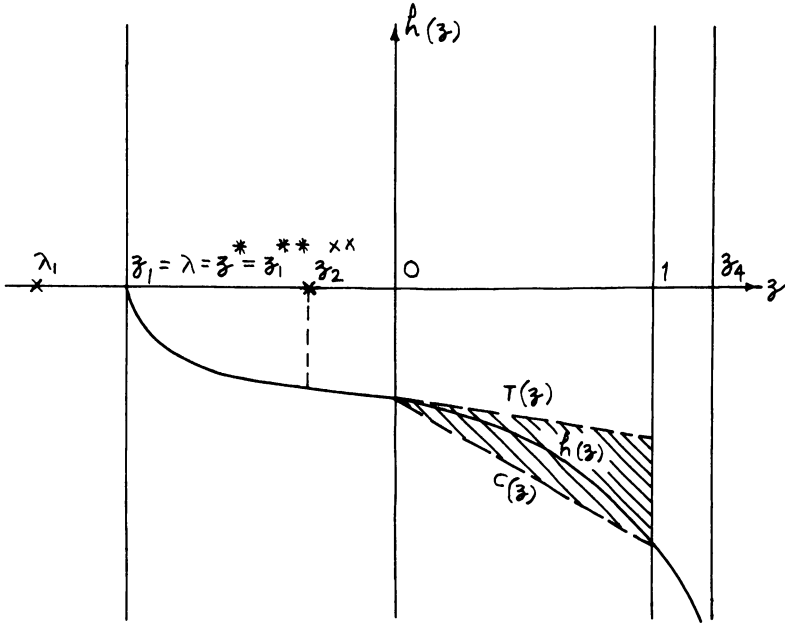

FIG. 2. $[a < 0; z_1 < \lambda < z_4]$.

FIG. 3. $[a < 0;\ \lambda_1 < \lambda = z_1 < z_4]$.

FIG. 4. $a < 0;\ \lambda_1 < \lambda < z_1 < z_4$.

FIG. 5. $[a < 0;\ \lambda < \lambda_1 < z_1 < z_4]$.
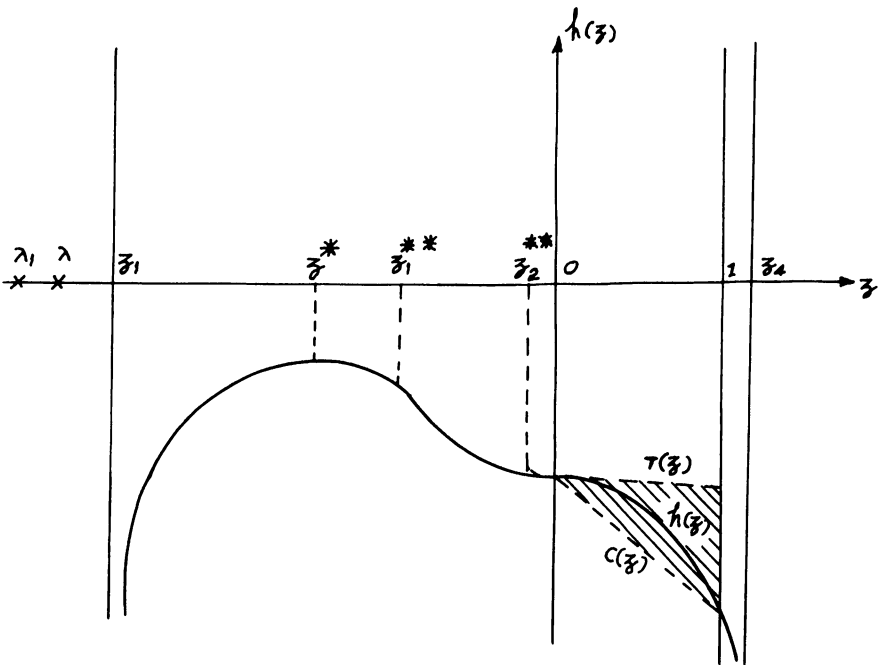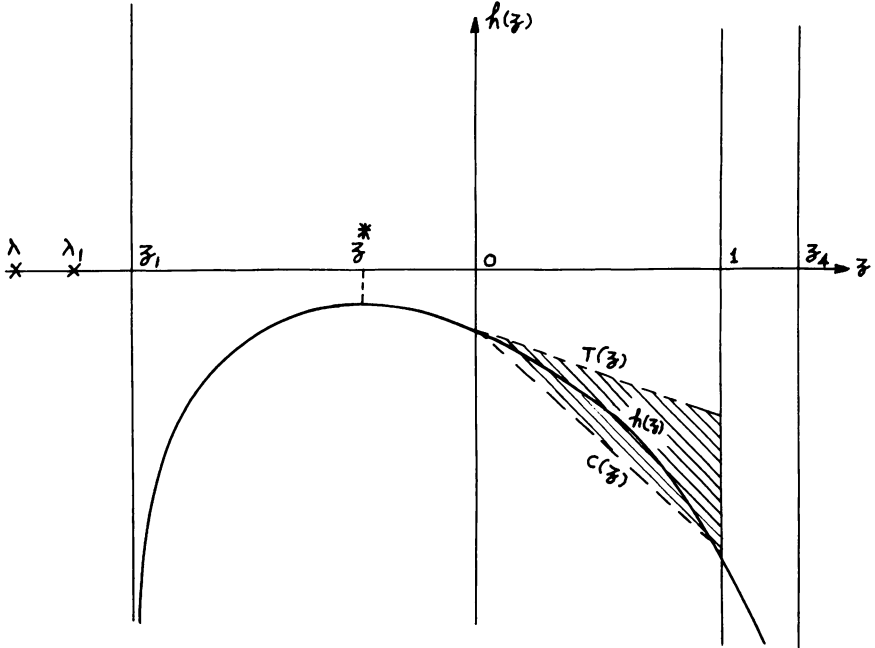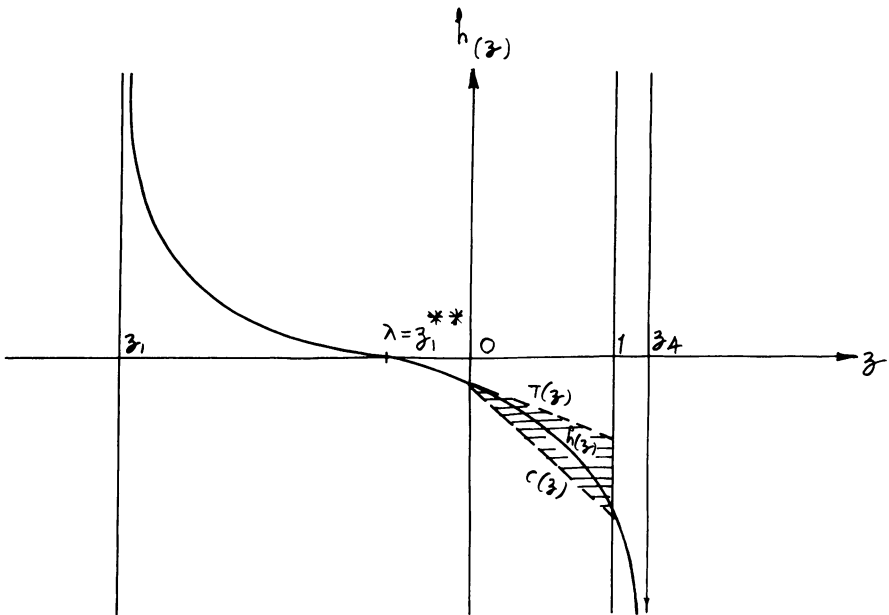

FIG. 6. $[a = 0;\ z_1 < \lambda = z_1^{**} < 0 < z_4]$.

Now we shall state the following propositions and prove them separately.

PROPOSITION I. Whenever $z_1 < \lambda < z_4$ the curve (3.2) can have only one point of inflexion and no stationary point (in the interval $z_1 < z < z_4$).

PROPOSITION II. When $\lambda_1 < \lambda < z_1$,

$$z_1 < z^* < z_1^{**} < z_2^{**} < 0 \tag{4.15}$$

provided $z_1 + z_4 \leq 0$.

PROPOSITION III. Defining

$$C(z) = h(0)(1 - z) + h(1)z \tag{4.16}$$

and

$$T(z) = h(1) + z \cdot dh(0)/dz \tag{4.17}$$

whenever

$$z_1 + z_4 \leq -1, \tag{4.18}$$

the inequalities

$$C(z) \leq h(z) \leq T(z) \tag{4.19}$$

are uniformly satisfied for $-\infty < \lambda \leq 2/3$ and $0 \leq z \leq 1$.

PROPOSITION IV. If $z_1 + z_4 \leq 0$, the inequalities

$$l(0) \leq l(z) \leq l(1) \tag{4.20}$$

are uniformly satisfied for $0 \leq z \leq 1$.

Proof of Proposition I. From (4.9) and (4.10) we have

$$\lambda_1 - z_1 = (z_4 - z_1)(4 - 3\sqrt{2})/8 < 0,$$
$$\lambda_2 - z_4 = (z_4 - z_1)(3\sqrt{2} - 4)/8 > 0, \tag{4.21}$$

so that $\lambda_1 < z_1 < z_4 < \lambda_2$.

Since points of inflexion z_i^{**} of $h(z)$ are real for $\lambda_1 \leq \lambda \leq \lambda_2$, they are a fortiori so for $z_1 < \lambda < z_4$ and are also distinct. But we shall prove that for $z_1 < \lambda < z_4$ there is only one point of inflection in the interval $z_1 < z < z_4$. We note that, due to (4.8),

$$2a(z_1 - z_1^{**})(z_1 - z_2^{**}) = 3(\lambda - z_1)(z_1 - z_4)^2/2$$

and

$$2a(z_4 - z_1^{**})(z_4 - z_2^{**}) = 3(\lambda - z_4)(z_1 - z_4)^2/2,$$

from which the following conditional inequalities follow:

$$z_1 < z_1^{**} < z_4 < z_2^{**}, \quad \text{when} \quad a > 0,$$
$$z_1^{**} < z_1 < z_2^{**} < z_4, \quad \text{when} \quad a < 0, \tag{4.22}$$

if we assume for definiteness that $z_1^{**} < z_2^{**}$ (see Figs. 1 and 2).

In the particular case when $a = 0$, again there is only one finite point of inflection given by (see Fig. 6)

$$z_1^{**} = \lambda = (z_1 + z_4)/2, \tag{4.23}$$

which also lies well within the limits under consideration. Thus from the above, it is clear that the curve of $h(z)$ can have only one point of inflection for $z_1 < \lambda < z_4$.

That the stationary point given by (4.7) does not occur in the interval $z_1 < z < z_4$ can be seen thus. Using (4.7), we obtain

$$z^* - z_1 = (z_4 - z_1)(\lambda - z_1)/a$$
$$z^* - z_4 = (z_4 - z_1)(z_4 - \lambda)/a \tag{4.24}$$

from which it follows that whenever $z_1 < \lambda < z_4$, $z^* < z_1$ or $z^* > z_4$ according as $a \lessgtr 0$.

This completes the proof of Proposition I.

Proof of Proposition II. When $\lambda_1 < \lambda < z_1$, there are two distinct points of inflection, let us say for definiteness

$$z_1^{**} = \{-c + (c^2 - 8ad)^{1/2}\}/4a \tag{4.25}$$

and

$$z_2^{**} = \{-c - (c^2 - 8ad)^{1/2}\}/4a. \tag{4.26}$$

Clearly

$$a = (\lambda - z_1) + (\lambda - z_4) < 0 \tag{4.27}$$

and if we assume $z_1 + z_4 < 0$, we have from (4.5) and (4.6)

$$2c = \{-a(z_1 + z_4) - 6\lambda(z_1 + z_4) + 12z_1z_4\} < 0$$

and $d < 0$ and therefore

$$z_1^{**} < z_2^{**} < 0. \tag{4.28}$$

Due to (4.27) and the first of (4.24) we have

$$z_1 < z^*. \tag{4.29}$$

Next, we shall prove that

$$z^* < z_1^{**}. \tag{4.30}$$

Substituting for λ_1 and λ_2 from (4.9) and (4.10), we have

$$(z_4 - z_1)^2 - 32(\lambda - \lambda_1)(\lambda_2 - \lambda) = 32(\lambda - z_1)(\lambda - z_4) > 0.$$

Obviously

$$(z_4 - z_1) > \{32(\lambda - \lambda_1)(\lambda_2 - \lambda)\}^{1/2}. \tag{4.31}$$

Hence

$$z^* - z_1^{**} = (z_4 - z_1)[(z_4 - z_1) - \{32(\lambda - \lambda_1)(\lambda_2 - \lambda)\}^{1/2}]/8a < 0,$$

as 'a' is negative due to (4.27). Hence (4.30) follows.

Combining (4.28) through (4.30), we have (4.25) (see Fig. 4) and this completes the proof of Proposition II.

It may be noted that in the specific cases when $\lambda = \lambda_1$ inequality (4.15) degenerates to

$$z_1 < z^* < z_1^{**} = z_2^{**} < 0$$

and to (see Fig. 3)

$$z_1 = z_1^* = z_1^{**} < z_2^{**} < 0$$

when $\lambda = z_1$. Further, $\lambda_1 \neq z_1$ in view of (4.21) and (2.13).

Proof of Proposition III. For the sake of convenience let us consider the following cases separately.

Case (i) $(-\infty < \lambda < \lambda_1)$. In this case, the curve of $h(z)$ lies completely below the z-axis and has only one stationary point occuring in the interval $z_1 < z < 0$ (see Fig. 5). When the condition (4.18) is satisfied it can be easily seen that a, b, c and d appearing in (4.3) to (4.6) are negative, and therefore from (4.1) and (4.2) we have

$$dh(z)/dz < 0,$$
$$d^2h(z)/dz^2 < 0 \tag{4.32}$$

in the interval $0 \leq z \leq 1$. Consequently $h(z)$ is a continuously decreasing convex function in the interval $0 \leq z \leq 1$ and therefore should lie between the chord joining $(0, h(0))$ and $(1, h(1))$ and the tangent at $(0, h(0))$ of (3.2).

Since (4.16) and (4.17) give respectively the ordinates of any point on the chord and the tangent mentioned above, in the interval $0 \leq z \leq 1$, we have (4.19) as required.

Case (ii) $(\lambda_1 \leq \lambda \leq z_1, \lambda_1 \neq z_1)$. It can be easily seen, in all the three cases under consideration, that (4.32) follows immediately from (4.11) and (4.12) due to Proposition II and the result is again (4.19).

Case (iii) $(z_1 < \lambda < 2/3)$. Due to Proposition I, the curve of $h(z)$ in the present case, has only one point of inflection and in general, either of the two conditional inequalities (4.22) holds.

First, when $z_1 < \lambda \leq 0$, (4.6) yields $d < 0$ due to (4.18), and therefore, from (4.8),

$$z_1^{**} \times z_2^{**} = d/2a,$$

has the opposite sign to a, that is, z_1^{**} and z_2^{**} have the same or opposite signs according as $a \lessgtr 0$. This together with (4.22) proves that (3.2) has no points of inflection in the interval $0 \leq z \leq 1$. Hence, using (4.22) to (4.24) in (4.11) and (4.12), we have again the inequalities (4.32). Thus the validity of (4.19) follows.

In particular, when $a = 0$, there is only one point of inflection given by (4.23), and it is negative due to (4.18) (see Fig. 6). The rest of the proof now follows straight away.

Secondly, when $0 < \lambda \leq 2/3$, we shall prove that for admissible values of z_1 and z_4 (see (2.13)) d is negative provided (4.18) is satisfied.

From (4.6), it is easily seen that d is negative if

$$\lambda < \bar{\lambda} = 4z_1z_4(z_1 + z_4)/\{3(z_1 + z_4)^2 - 4z_1z_4\}.$$

Presently we shall prove that $\bar{\lambda} > 2/3$ and therefore, for $0 < \lambda \leq 2/3$, $d < 0$ should follow.

When $z_1 + z_4 \leq -1$, to show $\bar{\lambda} > 2/3$, we may define an $\epsilon \geq 0$ such that $z_1 + z_4 = -(1 + \epsilon)$, and therefore

$$\bar{\lambda} - \tfrac{2}{3} = (\tfrac{2}{3}) [2z_1z_4\{3(z_1 + z_4) + 2\} - 3(z_1 + z_4)^2]/D_3$$

where D_3 is a positive expression.

As $z_4 > 1$, we further have

$$\bar{\lambda} - \tfrac{2}{3} > \{(\tfrac{2}{3}) (3\epsilon^2 + 8\epsilon + 1)/D_3\} > 0,$$

for all $\epsilon \geq 0$. This now enables us to conclude from (4.22) that the curve of $h(z)$ has no point of inflection in the interval $0 \leq z \leq 1$, for from (4.8) we have $z_1^{**} \cdot z_2^{**} = d/2a < 0$, as a is positive in the present case. From (4.24) we also have
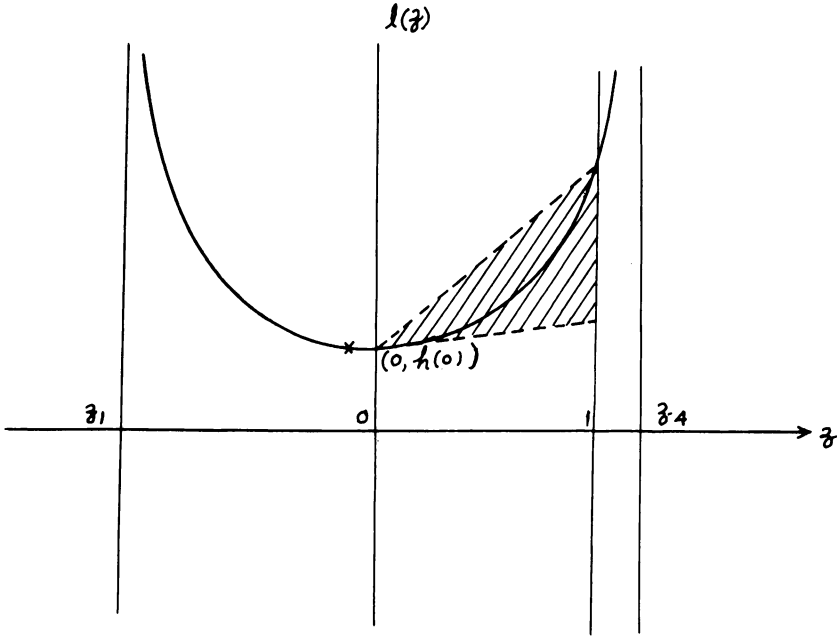
$$z^* > z_4. \tag{4.33}$$

FIG. 7.

Hence, in view of (4.33) and (4.22), (4.32) follows at once from (4.11) and (4.12). Consequently we have the inequality (4.19).

Combining all the several cases discussed above, the proof of proposition III follows. *Proof of Proposition IV.* From (3.6) we have

$$\{(z - z_1)(z_4 - z)\}^{3/2} \cdot dl(z)/dz = z - (z_1 + z_4)/2$$

and

$$\{(z - z_1)(z_4 - z)\}^{5/2} \cdot d^2l(z)/dz^2 = 2z^2 - 2(z_1 + z_4)z + \{(z_1 - z_4)^2 + 2(z_1 + z_4)^2\}/4$$

so that $dl(z)/dz = 0$ only for $z = (z_1 + z_4)/2 \leq 0$ and $d^2l(z)/dz^2 > 0$ in the interval $0 \leq z \leq 1$ as $z_1 + z_4 \leq 0$.

Therefore $l(z)$ is an increasing convex function in the interval $0 \leq z \leq 1$ and lies between the chord joining the two points $(0, l(0))$ and $(1, l(1))$ and the tangent at $(0, l(0))$, as shown in Fig. 7. Hence we have

$$l(0) + z(dl(0)/dz < l(z) < l(0)(1 - z) + l(1)z. \tag{4.34}$$

Since $z(d\, l(0)/dz) > 0$, we have

$$l(0) < l(0) + z(d\, l(0)/dz) \leq l(z), \text{ for } 0 \leq z \leq 1 \tag{4.35}$$

due to (4.34). Again, from (4.34)

$$l(z) \leq l(0) + \{l(1) - l(0)\}\, z \leq l(1), \text{ for } 0 \leq z \leq 1 \text{ as } l(1) > l(0). \tag{4.36}$$

Hence, combining (4.35) and (4.36), (4.20) follows.

This completes the proof of Proposition IV.

5. Auxiliary integrals.   For our subsequent references we shall consider the following integrals:

$$I_1 = \left(\frac{1}{2}\right) \int_{z_2}^{z_3} \{z_2 z_3/(z - z_2)(z_3 - z)\}^{1/2} \, dz/z, \tag{5.1}$$

$$I_2 = \left(\frac{1}{2}\right) \int_{z_2}^{z_3} \{(1 - z_2)(1 - z_3)/ (z - z_2)(z_3 - z)\}^{1/2} \, dz/(1 - z), \tag{5.2}$$

$$I_3 = \left(\frac{1}{2}\right) \int_{z_2}^{z_3} \{(z - z_2)(z_3 - z)\}^{-1/2} \, dz, \tag{5.3}$$

where z_2 and z_3 are such that $0 < z_2 < z_3 < 1$. Obviouslsy [2]

$$I_1 = I_2 = I_3 = \pi/2. \tag{5.4}$$

6. Bounds for the apsidal angle. The following theorems show that a Lock-Fowler missile has the same bounds as its equivalent common top whenever its launching conditions are restricted. The restriction is characterized by $z_1 + z_4 \leq -1$. It may be noted that in case of the common top $z_1 \to -\infty$ and this inequality is always satisfied for all finite values of z_4. Hence we claim that the Kohn-Hadamard [7,5] limits of a common top are independent of initial conditions.

THEOREM I. $- \pi < \Phi < -\pi/2$ whenever $z_1 + z_4 \leq 0$ and $- \infty < \lambda < 0$.
Proof. To obtain the lower bound for Φ we note from (3.1) that

$$\Phi = \frac{1}{2|\lambda|} \int_{z_2}^{z_3} \frac{(-z_1 z_2 z_3 z_4)^{1/2} \, h(z) \, dz}{z(1 - z) \{(z - z_2)(z_3 - z)\}^{1/2}}$$

where $\Omega > 0$, an assumption which can be made without loss of generality. By using Proposition III this becomes

$$\Phi > \frac{1}{2|\lambda|} \int_{z_2}^{z_3} \frac{(-z_1 z_2 z_3 z_4)^{1/2} C(z) \, dz}{z(1 - z) \{(z - z_2)(z_3 - z)\}^{1/2}} \quad .$$

Substituting for $C(z)$ from (4.16) in the above inequality with the values of $h(0)$ and $h(i)$ obtained from (3.2), we have

$$\Phi > -I_1 + \frac{(\lambda - 1)}{|\lambda|} \left\{ \frac{-z_1 z_2 z_3 z_4}{(1 - z_1)(1 - z_2)(1 - z_3)(z_4 - 1)} \right\}^{1/2} I_2$$

where I_1 and I_2 are as given in (5.1) and (5.2).

Using (2.22), this yeilds

$$\Phi > -I_1 - I_2 = -\pi. \tag{6.1}$$

Coming to the upper bound for Φ, we observe from (3.3) that

$$\Phi = - \Phi_1 - \Phi_2 \tag{6.2}$$

But by Proposition IV $l(z) \geq l(0)$ and therefore from (3.4) and (3.5) we have

$$\Phi_1 > I_1 = \pi/2$$

and

$$\Phi_2 > I_2\{(1 - z_1)(z_4 - 1)/(-z_1z_4)\}^{1/2} = (\pi/2)\{(1 - z_1)(z_4 - 1)/(-z_1z_4)\}^{1/2}$$

whence from (6.2) we get

$$\Phi < -(\pi/2)[1 + \{(1 - z_1)(z_4 - 1)/(-z_1z_4)\}^{1/2}] < -\pi/2 \qquad (6.3)$$

as

$$0 < \{(1 - z_1)(z_4 - 1)/(-z_1z_4)\}^{1/2} < 1.$$

Hence Theorem 1 follows from (6.1) and (6.3).

THEOREM 2. $-\pi/2 < \Phi < 0$ whenever $z_1 + z_4 \leq 1$ and $\lambda = 0$.

Proof. To prove the theorem, let us first consider the lower bound for Φ. When $\lambda = 0$, (3.3) yields

$$\Phi = -\Phi_2. \qquad (6.4)$$

Also, from (4.34), we have $l(z) \leq l(1)$, since $l(1) \geq l(0)$ whenever $z_1 + z_4 \leq 1$, and therefore from (3.5) $\Phi_2 < I_2 = \pi/2$.

Thus, it follows from (6.4) that

$$\Phi > -\pi/2.$$

For the upper bound we have plainly $\Phi < 0$, as the integrand appearing in Φ_2 (cf.(3.5)) is positive during the entire motion under consideration. Hence the theorem.

THEOREM 3. $0 < \Phi < \pi/2$ whenever $z_1 + z_4 \leq -1$ and $0 < \lambda \leq 2/3$.

Proof. To establish the lower bound we note from (3.3) that

$$\Phi = \Phi_1 - \Phi_2 \qquad (6.5)$$

where $\Phi_i (i = 1,2)$ are given by (3.4) and (3.5). Due to Proposition IV, it is easily seen that $\Phi_1 > \pi/2$ and $\Phi_2 < \pi/2$, whence from (6.5) we have $\Phi > 0$.

To get the upper bound, we note from (3.1) and Proposition III that

$$\Phi < \frac{1}{2\lambda} \int_{z_2}^{z_3} \frac{(-z_1z_2z_3z_4)^{1/2} \cdot T(z) \, dz}{z(1 - z)\{(z - z_2)(z_3 - z)\}^{1/2}}.$$

Substituting for $T(z)$ from (4.17) and simplifying, we have

$$\Phi < I_1 + I_2 \left\{ \frac{z_2z_3}{(1 - z_2)(1 - z_3)} \right\}^{1/2} \left\{ \frac{z_1 + z_4}{2z_1z_4} - \frac{(1 - \lambda)}{\lambda} \right\} = \frac{\pi}{2} (1 + f_1f_2) \qquad (6.6)$$

due to (2.22). In (6.6) f_1 and f_2 are given by

$$f_1 = \{(1 - z_1)(z_4 - 1)/(-z_1z_4)\}^{1/2}, \qquad (6.7)$$

$$f_2 = \{\lambda/2(1 - \lambda)\}(1/z_1 + 1/z_4) - 1 \qquad (6.8)$$

and are such that

$$0 < f_1 < 1, \qquad (6.9)$$

$$-1 < f_2 < 0, \qquad (6.10)$$

For, from (6.8) and (2.13) we obtain

$$f_2 < \frac{\lambda(1/z_1 + 1)}{2(1 - \lambda)} - 1 = \frac{\lambda}{2z_1(1 - \lambda)} + \frac{3(\lambda - 2/3)}{2(1 - \lambda)} < 0, \qquad (6.11)$$

since $0 < \lambda \le 2/3$, and other inequalities in (6.9) and (6.10) are evident.

Using (6.9) and (6.10) in (6.6), we have $\Phi < \pi/2$. This proves our assertion.

THEOREM 4. $\Phi > 0$ whenever $z_1 + z_4 \le 0$ and $2/3 < \lambda < 1$.

The proof is the same as in the previous case for the lower bounds; see also the next section.

THEOREM 5. $\Phi > \pi/2$ whenever $z_1 + z_4 \le 0$ and $\lambda \ge 1$.

Proof. When $\lambda = 1$, from (3.3) we have $\Phi = \Phi_1 > \pi/2$ (see (6.4)). When $\lambda > 1$, again from (3.3), we have

$$\Phi = \Phi_1 + \Phi_2$$

and that $\Phi_1 > \pi/2$ and $\Phi_2 > \pi f_1/2$ (see (6.4) and (6.5)). Hence, $\Phi > (1 + f_1)\pi/2$ and since $0 < f_1 < 1$, $\Phi > \pi/2$. Hence the Theorem.

7. Upper bounds when $\lambda > 2/3$. As to the question of finding an upper bound when $\lambda > 2/3$, the only answer is that there does not exist a sharp upper bound in such cases. However, by suitable majorization we may obtain the following upper bounds for Φ. If we assume that $z_1 + z_4 \le -2$, then by an obvious extension of Proposition III the inequalities (4.19) will be uniformly satisfied over the entire interval $-\infty < \lambda < 1$ and therefore in particular over the interval $2/3 < \lambda < 1$. Hence we have from (6.6)

$$\Phi < (1 + f_1 f_2)\pi/2.$$

By using (6.9) and (6.11), it follows that

$$\Phi < \frac{\pi}{2}\left\{1 + \frac{3(\lambda - 2/3)}{2(1 - \lambda)}\right\} = \frac{\pi}{4}\left(\frac{\lambda}{1 - \lambda}\right) \qquad (7.1)$$

When $\lambda = 1$, it may happen that $z_3 = z_4 = 1$; in this case we have from (3.3)

$$\Phi = \Phi_1 = \frac{1}{2}\int_{z_2}^{1}\left\{\frac{-z_1 z_2}{(z - z_1)(z - z_2)}\right\}^{1/2}\cdot\frac{dz}{z(1 - z)} = \infty \qquad (7.2)$$

and therefore a finite upper bound for Φ does not exist when $\lambda = 1$.

For $\lambda > 1$, we note from the nature of $h(z)$ that either $h(0)$ or $C(z)$ can be taken as an upper bound of $h(z)$ in $0 \le z \le 1$; i.e.,

$$h(z) \le h(0) \quad \text{for} \quad 1 < \lambda < z_4 \quad \text{and} \quad z_4 < \lambda < \lambda_2 \quad \text{when} \quad h(0) > h(1)$$

$$\le C(z) \quad \text{for} \quad \lambda_2 \le \lambda < \infty \quad \text{and} \quad z_4 < \lambda < \lambda_2 \quad \text{when} \quad h(0) < h(1).$$

When $h(z) \le C(z)$ we can prove that (see Theorem 1)

$$\Phi < \pi \qquad (7.3)$$

whereas when $h(z) \le h(0)$, we have directly from (3.3)

$$\Phi < \frac{\pi}{2} + \frac{\pi}{2}\{z_2 z_3/(1 - z_2)(1 - z_3)\}^{1/2} = \frac{\pi}{2}\{1 + \lambda f_1/(\lambda - 1)\},$$
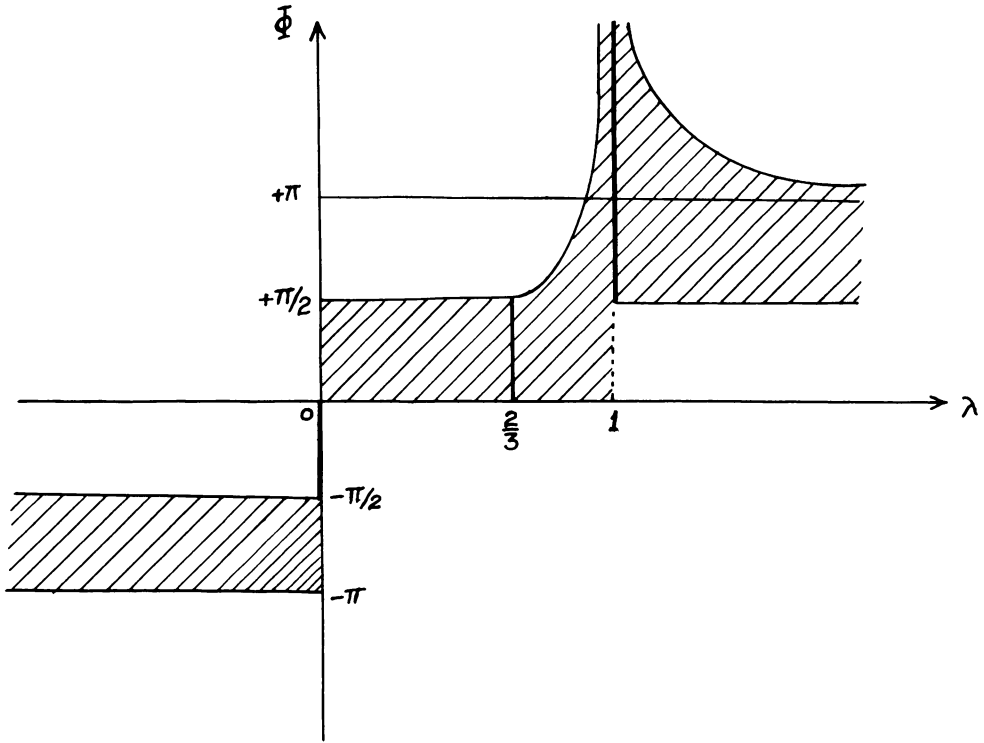
due to (2.22).

FIG. 8.

Now since $0 < f_1 < 1$ when $z_1 + z_4 < -2$, we get

$$\Phi < \frac{\pi}{2} + \pi\lambda/2(\lambda - 1) = \pi(2\lambda - 1)/2(\lambda - 1). \qquad (7.4)^*$$

When $\lambda > 1$, as the bounds given by (7.4) exceed those given by (7.3), we may select (7.4) as the upper bound for Φ over the interval $1 < \lambda < \infty$

It may be concluded here that all bounds of Φ we have established in Secs. 6 and 7 are valid without a single exception provided $z_1 + z_4 \leq -2$. The various bounds of Φ are shown in Fig. 8. For this, also refer to [4] and [7], the parameter r used in [4] and [7] is related to λ (given here) through $2\lambda = 1 + r$.

8. On Hadamard's theorem. The Halphen-Hadamard theorem [5, 6] on a heavy symmetrical top states that the total advance in azimuth of the top describing loops has the same sense as its precession on the lowest level. The mathematical analogue of this theorem in our case is

$$\Phi > 0, \qquad z_2 < \lambda < z_3. \qquad (8.1)$$

In the present system, the result (8.1) is not valid. This may be seen from the following example.

If we choose $\Omega = 1$, $q = 3.69$, $s = 2$, $z_0 = 0.15$, $z_0' = 0$ and $\phi_0' = 0.392$, we have $\lambda = 2z_0(1 - z_0) + z_0 = 0.25$ and

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*(7.4) is true when $z_1 + z_4 \leq 0$.

$$E = 4z_0(1 - z_0)(\phi_0'^2 - q) + \left(\frac{1}{2s} - 2q\right)(2z_0 - 1) + q = 6.884.$$

Now Eqs. (2.8) and (2.9) are given by

$$z_1'^2 = 14.72(z - z_1)(z - z_2)(z - z_3)(z - z_4) \tag{8.2}$$

where $z_1 = -0.016$, $z_2(= z_0) = 0.150$, $z_3 = 0.950$, $z_4 = 1.882$ and

$$\phi' = \frac{0.25 - z}{2z(1 - z)}. \tag{8.3}$$

We have, as before, $\Phi = \Phi_1 - \Phi_2$ and these elliptic integrals Φ_i may be evaluated thus [1, pp. 113 and 225]:

$$\Phi_1 = 0.033 \int_{0.15}^{0.95} \frac{dz}{z\{(z - z_1)(z - z_2)(z_3 - z)(z_4 - z)\}^{1/2}}.$$

Using the transformation

$$Sn^2u = (z_3 - z_1)(z - z_2)/(z_3 - z_2)(z - z_1)$$

we have

$$\Phi_1 = 0.34 \int_0^{2.616} \frac{1 - 0.828Sn^2u}{1 + 0.087Sn^2u}\, du = 0.403$$

and likewise [1, pp. 115 and 228]

$$\Phi_2 = 0.177 \int_0^{2.616} \frac{1 - 0.828Sn^2u}{1 + 0.990Sn^2u}\, du = 1.663.$$

Now clearly we have $\Phi = -1.26 < 0$.

If we can put an additional restriction by which we limit our choice of initial conditions we may state an analogue of the previous result by the following theorem.

THEOREM 6. If $z_1 + z_4 \leq 0$, we have $\Phi > 0$, whenever $z_2 < \lambda < z_3$.

The proof of this result is already implied in Theorems 3 and 4.

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