TRANSIENT RESPONSE OF AN ACOUSTIC HALF-SPACE TO A ROTATING POINT LOAD*

By

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Abstract. This paper presents a solution of three-dimensional problems for nonuniformly moving loads. By using Cagniard's technique, the problem is reduced to a change in the order of integration. The technique has no restrictions on the velocity variation and on the trajectory of the load.

The problem of a rotating load applied to the surface of an acoustic half-space is discussed as an example. It is found that the leading wave turns on the surface of a hyperboloid of one sheet of revolution as the load rotates supersonically. There exists singularity just ahead of the leading wave-front, which, however, does not change order due to the rotation.

1. Introduction. The problem of determining the transient response of a body subjected to a moving load has long been of interest. For such problems Cagniard's technique [1] is useful because of the simplicity of the Laplace inversion and of the wave analysis. Thus, the technique has been applied usefully to the problem of a uniformly moving load [2, 3, 4]. Nevertheless, it seems that there has been no direct application of the technique to the problem of a non-uniformly moving load. Recently, Freund [5, 6] has developed an analytical technique for the problem of this type, and in previous papers [7, 8] the author solved one of the two-dimensional problems of a reciprocating line load.

The present paper is a generalization of Freund's technique, as originally developed for two-dimensional problems, to three-dimensional problems of a non-uniformly moving load. It imposes no restrictions on either the velocity variation or the trajectory of the load, and reduces the problem to a change in the order of integration to obtain the final Laplace inversion by inspection.

The problem of a point load rotating on the surface of an acoustic half-space is discussed as an example. It is found that the leading wave turns on the surface of a hyperboloid of one sheet of revolution as the load rotates supersonically. There exists a singularity ahead of the leading front; however, the singularity does not change order due to the rotation.

2. Method of solution. Let us consider the acoustic half-space depicted in Fig. 1; $z > 0$ forms the interior and $z = 0$ is the surface. Assuming that the space is a non-viscous and

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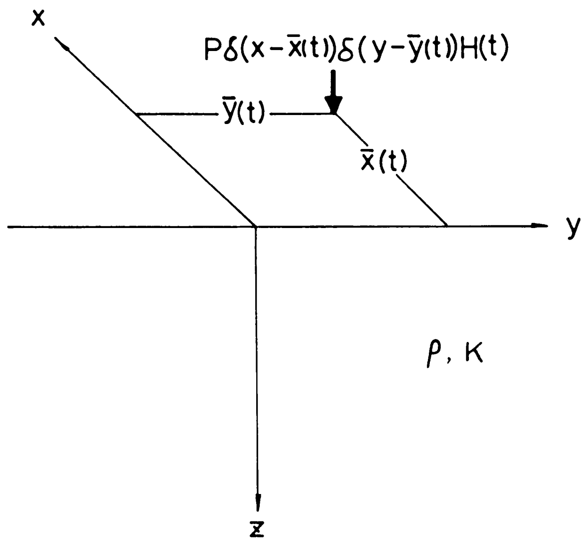


FIG. 1. Coordinate system.

incompressible fluid, the displacement potential ϕ satisfies the wave equation

$$\phi_{,ii} = c^{-2}\phi_{,tt}, \qquad c = (K/\rho)^{1/2}, \qquad i = x, y, z \tag{1}$$

where ρ, K and c are density, bulk modulus and velocity of the acoustic wave, respectively. The displacement U_i and the pressure p are given by

$$U_i = \phi_{,i}, \qquad p = -\rho\phi_{,tt} \quad . \tag{2}$$

A point load of magnitude P is suddenly applied and subsequently moves on the surface; the position of the load is given by $(\bar{x}(t), \bar{y}(t), 0)$ and the boundary condition on the surface is thus given by

$$p|_{z=0} = P\delta(x - \bar{x}(t))\delta(y - \bar{y}(t))H(t) \tag{3}$$

It is assumed that the quiescent condition at $t = 0$ is

$$U_i|_{t=0} = U_{i,t}|_{t=0} = 0. \tag{4}$$

Then, in order to illustrate our technique for the problem just stated, we apply Laplace and Fourier transforms, defined by

$$f^*(s) = \int_0^\infty f(t) \exp(-st)\, dt, \tag{5}$$

$$f(\xi, \eta) = \iint_{-\infty}^{\infty} f(x,y) \exp(i\zeta x + i\eta y) dx\, dy \ , \tag{6}$$

$$f(x,y) = (2\pi)^{-2} \iint_{-\infty}^{\infty} \bar{\bar{f}}(\xi, \eta) \exp(-i\xi x - i\eta y)\, d\xi\, d\eta, \tag{7}$$

to Eqs. (1–3). Following the procedure in references [7, 8], we put the transformed Eq. (3) in the form

$$\bar{p}^*|_{z=0} = P\int_0^\infty \exp\{i\xi x(t') + i\eta y(t') - st'\}\, dt'. \tag{8}$$

Thus, the potential is given by

$$\bar{\phi}^* = -P(\rho s^2)^{-1}\int_0^\infty \exp\{i\xi x(t') + i\eta y(t') - \alpha z - st'\}\, dt', \tag{9}$$

where

$$\alpha = (\xi^2 + \eta^2 + s^2/c^2)^{1/2}, \quad \operatorname{Re}(\alpha) \geq 0. \tag{10}$$

Next, in order to apply Cagniard's technique, we introduce the following transformations:

$$X = x/l, \qquad Y = y/l, \qquad Z = z/l, \qquad \zeta = ct'/l, \tag{11}$$

$$\bar{X}(\zeta) = \bar{x}(t')/l, \qquad \bar{Y}(\zeta) = \bar{y}(t')/l, \tag{12}$$

$$\xi = s(cJ)^{-1}[\{X - \bar{X}(\zeta)\}u - \{Y - \bar{Y}(\zeta)\}v], \tag{13}$$

$$\eta = s(cJ)^{-1}[\{X - \bar{X}(\zeta)\}v + \{Y - \bar{Y}(\zeta)\}u], \tag{14}$$

$$J = [\{X - \bar{X}(\zeta)\}^2 + \{Y - \bar{Y}(\zeta)\}^2]^{1/2}, \tag{15}$$

where l is a representative length. The transformation of Eqs. (13–14) is similar to that used by Gakenheimer and Miklowitz [2]. The Fourier inversion of Eq. (9), hence, can be converted to

$$\phi^* = -l\mathrm{P}\,(2\pi^2\rho c^3)^{-1}\int_0^\infty d\zeta \int_0^\infty dv \int_{-\infty}^\infty \exp\{-sl(iJu + Z\,(u^2 + v^2 + 1)^{1/2} + \zeta)/c\}\, du. \tag{16}$$

Further, application of Cagniard's technique transforms Eq. (16) into

$$\phi^* = -P(2\pi^2K)^{-1}\int_0^\infty d\zeta \int_0^\infty dv \int_{l\{\zeta + ((v^2+1)(J^2+Z^2))^{1/2}\}/c}^\infty \exp(-st)\,\{du^+/d\tau - du^-/d\tau\}\, dt \tag{17}$$

where

$$u^\pm = (J^2 + Z^2)^{-1}\{-iJ(\tau - \zeta) \pm Z\,((\tau - \zeta)^2 - (v^2 + 1)(J^2 + Z^2))^{1/2}\}, \tag{18}$$

$$\tau = ct/l, \tag{19}$$

and after changing the order of integration the evaluation of the integral with respect to v yields

$$\phi^* = -P(2\pi K)^{-1}\int_0^\infty d\zeta \int_{lT(\zeta)/c}^\infty F(\zeta; \tau, X, Y, Z)\exp(-st)\, dt, \tag{20}$$

where

$$F(\zeta; \tau, X, Y, Z) = Z(\tau - \zeta)\,[\{X - \bar{X}(\zeta)\}^2 + \{Y - \bar{Y}(\zeta)\}^2 + Z^2]^{-3/2}, \tag{21}$$

$$T(\zeta) = \zeta + [\{X - \bar{X}(\zeta)\}^2 + \{Y - \bar{Y}(\zeta)\}^2 + Z^2]^{1/2}\quad . \tag{22}$$

Note that the inner integral in Eq. (20) is the same as the Laplace transform defined by Eq. (5). Thus, the Laplace inversion of Eq. (20) can be obtained by inspection after changing the order of integration, and the region of support for the double integration is governed by Eq. (22). So, considering Eq. (22), we find that $T(\zeta)$ is the arrival time at point (X, Y, Z) of a wave emanating from a loaded point $(\bar{X}(\zeta), \bar{Y}(\zeta), 0)$ at time ζ. Therefore, $T(\zeta)$ should be called the "arrival time function" and the Laplace inversion of the problem is reduced to an exchange in the order of integration for a given load motion $\bar{X}(\zeta)$, $\bar{Y}(\zeta)$.

3. Example: a rotating load. Let us consider a load rotating on a circle l with angular velocity ω. Then the position of the load is given in non-dimensional form as

$$\bar{X}(\zeta) = \cos(\Omega\zeta), \qquad \bar{Y}(\zeta) = \sin(\Omega\zeta), \qquad \Omega = l\omega/c \tag{23}$$

and the arrival time function is

$$T(\zeta) = \zeta + (\rho^2 - 2\rho\cos(\Omega\zeta - \theta) + 1 + Z^2)^{1/2}, \tag{24}$$

where the non-dimensional cylindrical coordinate system (ρ, θ, Z) is introduced as

$$\rho = (X^2 + Y^2)^{1/2}, \qquad \theta = \tan^{-1}(Y/X), \qquad 0 \le \theta < 2\pi \quad . \tag{25}$$

We conclude from Eq. (24) that:

(A) if $1/\Omega \ge \{((\rho + 1)^2 + Z_2)^{1/2} - ((\rho - 1)^2 + Z^2)^{1/2}\}/2$, $T(\zeta)$ is a monotonically increasing function;

(B) if $1/\Omega < \{(\rho + 1)^2 + Z^2)^{1/2} - ((\rho - 1)^2 + Z^2)^{1/2}\}/2$, $T(\zeta)$ takes maxima and minima at $\zeta = \zeta_j^{(1)}$ and $\zeta_j^{(2)}$, respectively, where

$$\zeta_j^{(1)} = (\theta + 2\pi j - \psi)/\Omega \,, \qquad \zeta_j^{(2)} = (\theta + 2\pi j - \tilde{\psi})/\Omega \quad , \tag{26}$$

$$\psi = \cos^{-1}[(\rho\Omega^2)^{-1}\{1 - (1 + \Omega^2(\rho^2\Omega^2 - \rho^2 - Z^2 - 1))^{1/2}\}] \tag{27}$$

$$\tilde{\psi} = \cos^{-1}[(\rho\Omega^2)^{-1}\{1 + (1 + \Omega^2(\rho^2\Omega^2 - \rho^2 - Z^2 - 1))^{1/2}\}] \tag{28}$$

where $0 < \cos^{-1}(\ ) < \pi$, $j = 0, 1, 2 \cdots$. It is clear that condition (A) is automatically satisfied at all points in the entire space if the load rotates subsonically ($\Omega \le 1$). However, in the supersonic case ($\Omega > 1$), condition (A) is satisfied at the observation points (x, y, z) in the sub-region

$$Z \ge ((\rho^2\Omega^2 - 1)(\Omega^2 - 1))^{1/2}/\Omega, \tag{29}$$

and condition (B) is satisfied in its outer region.

Consider the inversion of Eq. (20) in case (A). As $T(\zeta)$ is a monotonically increasing function, its inverse function, defined by $\zeta = T^{-1}(\tau)$, is also a single-valued function. Thus the region of support for the double integrations in Eq. (20) is as given in Fig. 2. The exchange of the order of integration in this case is very easy and yields

$$\phi^* = -P(2\pi K)^{-1}\int_{lT(0)/c}^{\infty} \exp(-st)\,dt \int_0^{T^{-1}(\tau)} F(\zeta; \tau, \rho, \theta, Z)\,d\zeta; \tag{30}$$

further, by inspection, its inverse is

$$\phi = -P(2\pi K)^{-1}H(\tau - (\rho^2 - 2\rho\cos\theta + 1 + Z^2)^{1/2})\int_0^{T^{-1}(\tau)} F(\zeta; \tau, \rho, \theta, Z)\,d\zeta \tag{31}$$

where $H(\ )$ is the Heaviside unit step function and

$$T(0) = (\rho^2 - 2\rho\cos\theta + 1 + z^2)^{1/2} \quad , \tag{32}$$

$$F(\zeta \cdot \tau, \rho, \theta, Z) = Z(\tau - \zeta)\{\rho^2 - 2\rho\cos(\Omega\zeta - \theta) + 1 + Z^2\}^{-3/2} \quad . \tag{33}$$

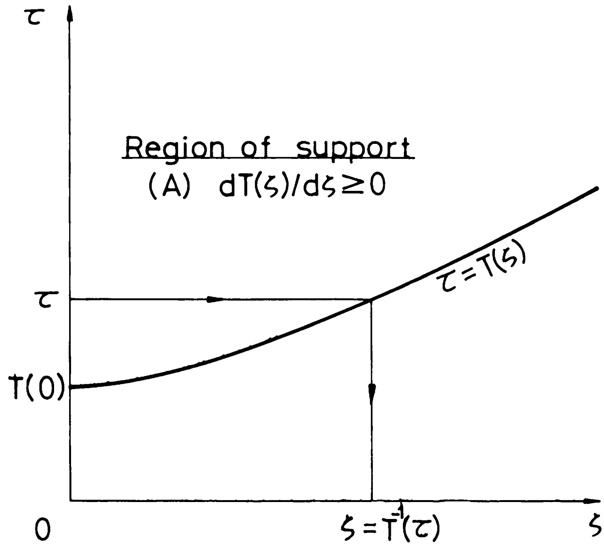


FIG. 2. The region of support for the double integrations of Eq. (20) for case (A).

On the other hand, case (B) is decomposed into the following three cases depending on the angle θ. That is,

$$\text{(B1):}\quad 0 \le \theta < \tilde{\psi},$$
$$\text{(B2):}\quad \tilde{\psi} \le \theta < \psi,$$
$$\text{(B3):}\quad \psi \le \theta < 2\pi.$$

The region of support for the double integrations is as shown in Fig. 3 for case (B1). We divide the region into R_n ($n = 1, 2, 3, \cdots$.) bounded with the maxima of the arrival time function as shown in the figure, and define an inverse function for each domain as

$$\left.\begin{matrix} \eta_j^{(1)} \\ \eta_j^{(2)} \end{matrix}\right\} = T^{-1}(\tau) \qquad \begin{matrix} \zeta_{j-1}^{(2)} < \eta_j^{(1)} < \zeta_j^{(1)} \\ \zeta_j^{(1)} < \eta_j^{(2)} < \zeta_j^{(2)} \end{matrix} \tag{34}$$

Then the decomposition of the region changes Eq. (20) to the alternative form

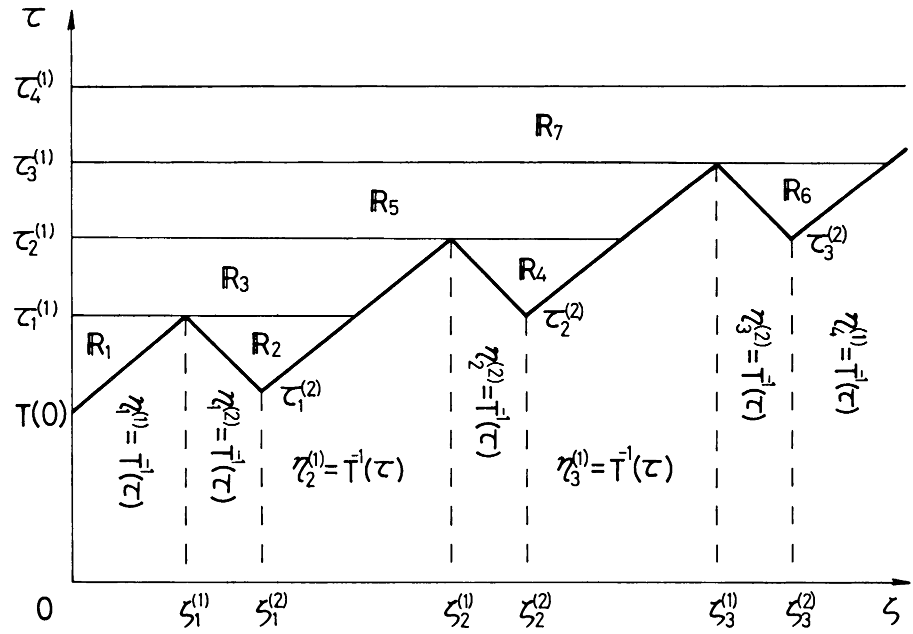
$$\phi^* = -P(2\pi K)^{-1} \sum_{n=N}^{\infty} \iint_{R_n} \exp(-st) F(\zeta;\, \tau, \rho, \theta, Z)\, d\zeta\, dt \quad , \tag{35}$$

where

$$\begin{aligned} N &= 1\,;\quad 0 \le \theta < \tilde{\psi} \\ &= 0\,;\quad \tilde{\psi} \le \theta < \psi \\ &= -1\,;\quad \psi \le \theta < 2\pi \end{aligned}$$

and the order of integration is changed as follows:

$$\iint_{R_{2j+1}} \exp(-st)\, F(\zeta;\, \tau, \rho, \theta, Z)\, d\zeta\, dt$$
$$= \int_{l\tau_{j-1}^{(1)}/c}^{l\tau_j^{(1)}/c} \exp(-st)\, dt \int_0^{\eta_j^{(1)}} F(\zeta;\, \tau, \rho, \theta, Z)\, d\zeta \quad , \tag{36}$$



(a) $0 \leq \theta < \tilde{\Psi}$

FIG. 3. The region of support for the double integrations of Eq. (20) for case (B1).

$$\iint_{R_{2j}} \exp(-st) F(\zeta;\, \tau, \rho, \theta, Z)\, d\zeta\, dt$$

$$= \int_{l\tau_j^{(2)}/c}^{l\tau_j^{(1)}/c} \exp(-st)\, dt \int_{\eta_j^{(2)}}^{\eta_{j+1}^{(1)}} F(\zeta;\, \tau, \rho, \theta, z)\, d\zeta \quad , \tag{37}$$

where

$$\tau_j^{(1)} = T(\zeta_j^{(1)}), \qquad \tau_j^{(2)} = T(\zeta_j^{(2)}), \qquad j = 0, 1, 2, 3, \cdots . \tag{38}$$

The right-hand side of Eq. (35) is now the expression for a Laplace transform and the inversion is thus easily obtained by inspection:

$$\phi = -P(2\pi K)^{-1} \left\{ \sum_{j=N_1}^{\infty} H(\tau - \tau_{j-1}^{(1)}) H(\tau_j^{(1)} - \tau) \int_0^{\eta_j^{(1)}} F(\zeta;\, \tau, \rho, \theta, Z)\, d\zeta \right.$$

$$\left. + \sum_{j=N_2}^{\infty} H(\tau - \tau_j^{(2)}) H(\tau_j^{(1)} - \tau) \int_{\eta_j^{(2)}}^{\eta_j^{(1)}} F(\zeta;\, \tau, \rho, \theta, Z)\, d\zeta \right. \quad ,$$

where

(B1): $N_1 = N_2 = 1, \quad \tau_0^{(1)} = T(0), \quad \zeta_0^{(2)} = 0,$

(B2): $N_1 = 1, N_2 = 0, \quad \tau_0^{(1)} = T(0), \quad \zeta_0^{(1)} = 0,$

(B3): $N_1 = N_2 = 0, \quad \tau_{-1}^{(1)} = T(0), \quad \zeta_{-1}^{(2)} = 0$.

Consequently, the potential ϕ is given by Eqs. (35) and (39) in the cases (A) and (B), respectively. The pressure p can be derived from Eq. (2) and yields

$$2\pi l^2 P^{-1} p = H(\tau - (\rho^2 - 2\rho\cos\theta + 1 + Z^2)^{1/2})\, G(T^{-1}(\tau)) \quad \text{(case (A))}$$

$$= \sum_{j=N_1}^{\infty} H(\tau - \tau_{j-1}^{(1)}) H(\tau_j^{(1)} - \tau) G(\eta_j^{(1)})$$

$$+ \sum_{j=N_2}^{\infty} H(\tau - \tau_j^{(2)}) H(\tau_j^{(1)} - \tau)\{G(\eta_{j+1}^{(1)}) - G(\eta_j^{(2)})\} \quad \text{(case (B))} \tag{40}$$

where

$$G(\eta) = Z(\tau - \eta)^{-2}\{d^2\eta/d\tau^2 + 2(\tau - \eta)^{-1}(d\eta/d\tau)^2 - (\tau - \eta)^{-1}\, d\eta/d\tau\} \quad . \tag{41}$$

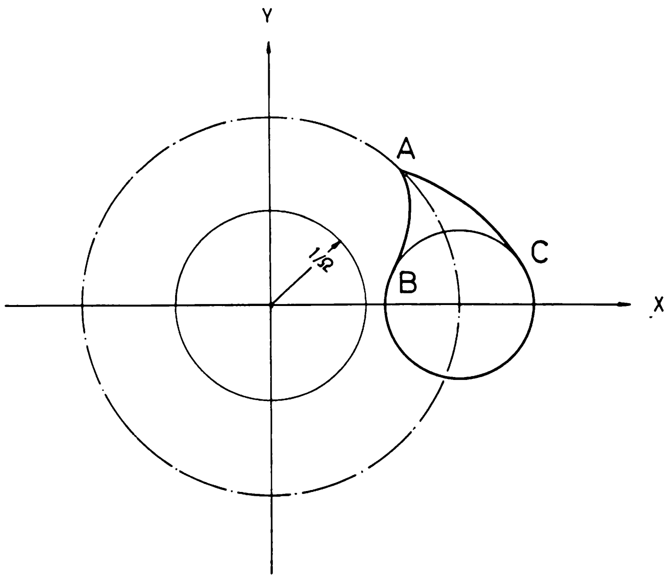
4. Wave analysis. It is clear that there is only one hemispherical wave pattern which is centered at $(1, 0, 0)$ and is given by $\tau = T(0) = (\rho^2 - 2\rho\cos\theta + 1 + Z^2)^{1/2}$ in the subsonic case $\Omega \leq 1$. But in the supersonic case $\Omega > 1$ there appears a leading wave. Computing the sets of the observation points where the arrival time function takes minima and maxima for a given time, we can find that the former set shows the ordinary leading wavefront and the latter shows the rotated leading wavefront. These patterns on the surface of the space are shown in Fig. 4 where the rotated front is the arc AB′. The condition that the arrival time function takes stationary values shows us that the begining of the rotation is

$$\tau = (1 - 1/\Omega^2)^{1/2} \tag{42}$$

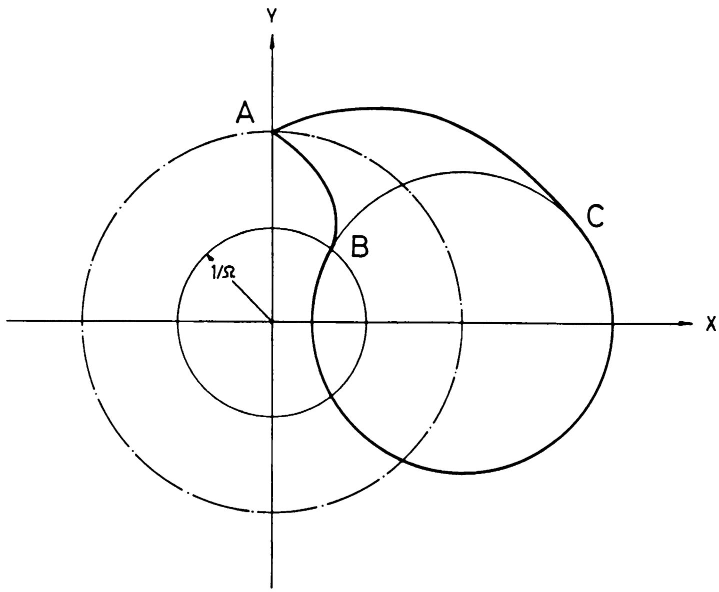
and that the rotation takes place on the surface of a hyperboloid of one sheet of revolution,

$$Z = ((\rho^2\Omega^2 - 1)(\Omega^2 - 1))^{1/2}/\Omega \quad . \tag{43}$$

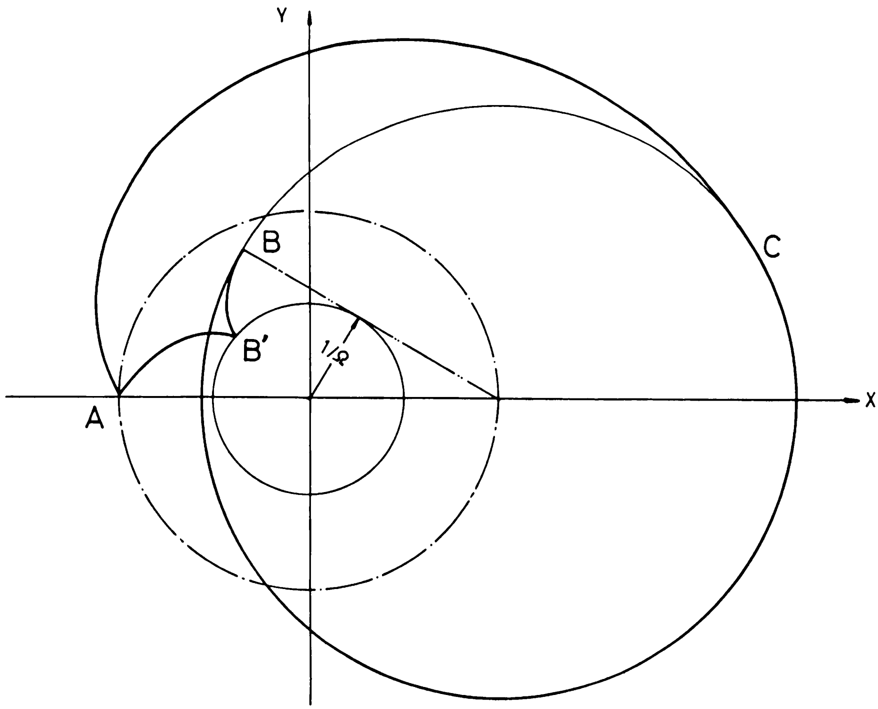
Expanding Eq. (40) in the neighborhood of $\tau = T(0)$, $\tau_j^{(1)}$ and $\tau_j^{(2)}$, we can get the



(a) $\tau = \pi/8$ · $\Omega = 2$



(b) $\tau = \pi/4$, $\Omega = 2$



(c) $\tau = \pi/2$, $\Omega = 2$

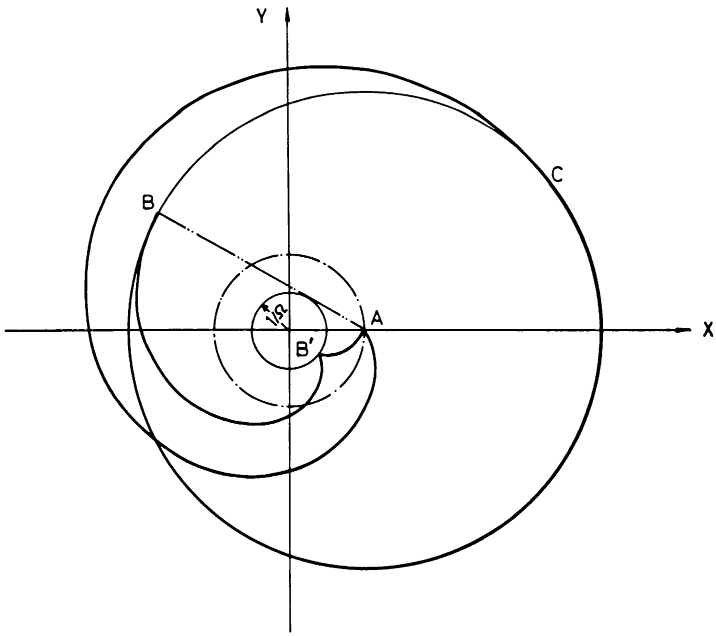


FIG. 4. Wave patterns on the surface for the supersonic case $\Omega = 2$: (a) $\tau = \pi/8$, (b) $\tau = \pi/4$, (c) $\tau = \pi/2$, (d) $\tau = \pi$.

wavefront singularity of the pressure as

$$\begin{aligned} &\simeq O(1); \qquad \tau \to (\rho^2 - 2\rho\cos\theta + 1 + Z^2)^{1/2} \\ 2\pi l^2 P^{-1} p &\simeq \{A_j^{(1)}(\tau_j^{(1)} - \tau)^{-3/2} + O\{(\tau_j^{(1)} - \tau)^{-1/2}\}; \quad \tau \to \tau_j^{(1)} \\ &\simeq A_j^{(2)}(\tau - \tau_j^{(2)})^{-3/2} + O\{(\tau - \tau_j^{(2)})^{-1/2}\}; \; \tau \to \tau_j^{(2)} \end{aligned} \tag{44}$$

where

$$A_j^{(i)} = Z[2\{\tau_j^{(i)} - \zeta_j^{(i)}\}(2\,|M(\zeta_j^{(i)})\,|)^{1/2}]^{-1}; \qquad i = 1, 2 \quad , \tag{45}$$

$$\begin{aligned} M(\zeta) = \rho\Omega^2[\cos(\Omega\zeta - \theta) - \rho\sin^2(\Omega\zeta - \theta)\{\rho^2 - 2\rho\cos(\Omega\zeta - \theta) + 1 + Z^2\}^{-1}] \\ \times \{\rho^2 - 2\rho\cos(\Omega\zeta - \theta) + 1 + Z^2\}^{-1/2}. \end{aligned} \tag{46}$$

It is noteworthy that singularity appears just ahead of the leading front, but that the singularity does not change its order due to the rotation.

5. Conclusion. The technique developed here consists in reducing the problem of a moving load to a change in the order of integration, and reduces the analysis of wave to a simple consideration of the arrival time function. As a result, it is noted that a set of observation points, where the arrival time function takes an extremum, constitutes a characteristic wavefront. Moreover, as the technique has no restrictions on the velocity variation and on the trajectory of the load, we can conclude that the present technique is an generalization of Freund's [5] to the three-dimensional problem of a nonuniformly moving load.

The problem of a rotating load on the surface of an acoustic half-space is discussed as an example. It is found that the rotation of the leading wave takes place on the surface of a hyperboloid of one sheet of revolution when the load rotates supersonically.

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