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CURVES ALONG WHICH PLANE WAVES CAN INTERFERE*

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Summary. Partial results are given on a conjecture in inverse scattering theory concerning the interference of two-dimensional plane waves. The conjecture states that an odd number of plane waves of the same frequency can only cancel each other at isolated points and not along a simple continuous curve. It is partially confirmed here for curves which are nearly flat at some point. An analysis is also made for various possible nodes for an even number of plane waves.

1. Introduction. In [1] Karp treats the inverse scattering problem in two dimensions. He shows that a crucial role is played by the plane wave far-field scattering amplitude. The latter is defined by

$$f(\theta, \theta_0) = \lim_{r\to\infty} (\pi kr/2)^{1/2} V(r, \theta; \theta_0) \exp\left(-i\left(kr - \frac{\pi}{4}\right)\right), \tag{1.1}$$

where $V(r, \theta; \theta_0)$ is the outgoing scattered field at (r, θ) due to an incoming plane wave

$$\varphi(r, \theta; \theta_0) = \exp(ikr \cos(\theta - \theta_0)) \tag{1.2}$$

in the direction θ_0, incident on a scatterer S. The inverse scattering problem is the problem of determining the obstacle S from $f(\theta, \theta_0)$. Karp proved for Dirichlet boundary conditions that if $\theta_1, \theta_2, \cdots, \theta_n$ are distinct angles and the determinant $\det f(\theta_i, \theta_j) = 0$, while no subdeterminant vanishes, then S lies on the locus

$$\sum_{j=1}^{n} A_j \exp[ik(x \cos\theta_j + y \sin\theta_j)] = 0$$

for appropriate complex non-zero $A_1, A_2, \cdots, A_n$. Hence S is restricted to being a node, i.e. a curve of interference, of n plane waves of the same frequency. He also showed that

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three plane waves can never interfere along any smooth arc, and he conjectured that this is the case for any odd n. This would mean that for odd n $\det f(\theta_i, \theta_j)$ never vanishes (if no subdeterminant does).

The precise, somewhat more general statement of the conjecture is: if $\theta_1, \theta_2, \cdots, \theta_n$ are distinct angles (modulo 2π) and

$$\sum_{j=1}^{n} A_j \exp(ik(x \cos\theta_j + y \sin\theta_j)) = 0 \tag{1.3}$$

along some simple continuous arc S given by $y = g(x)$, $a \leq x \leq b$, where the A_j are complex non-zero, and $k > 0$, then n is even. A corollary of the conjecture is that for the Dirichlet problem it is impossible to avoid having a scattered field when the incident field consists of an odd number of plane waves. Conversely, it is known that if the incident field is zero for a smooth finite arc then the field scattered off the arc is zero [3].

Although we have not settled the conjecture, we have obtained some results on the possible shapes of the arc S. In Sec. 2 we discuss various possible curves along which an even number of waves can interfere and exhibit a technique for generating such nodes. We explain in Sec. 3 how the problem of settling the conjecture can be simplified by rotation and translation of coordinates. In Secs. 4 and 5 we establish the conjecture for the cases $n = 3$ and S a line (or line segment), respectively. These results are generalized in Sec. 6 to the case where S is "almost" a linear scatterer. We explain in Sec. 7 the difficulties encountered in attempting to "force" a decision on the conjecture by the use of general methods.

2. Some simple results for n even. It is clear that any number of plane waves can be made to interfere at any specified isolated point by choosing appropriate amplitudes A_j. If n waves interfere along the arc S, always assumed simple and continuous, we will say that they possess the scatterer S. (Hence the conjecture asserts that no odd number of waves can possess a scatterer.)

The following facts are easy to show:

(a) Two waves can only possess a linear scatterer (a line or line segment(s)) which bisects the angle formed by the corresponding rays (Fig. 1).

(b) For any even $n = 2m$ there are n waves which possess a scatterer, in fact a linear scatterer. Indeed, choose $k > 0$ and angles θ_j such that $0 < \theta_1 < \theta_2 < \cdots < \theta m < \pi/2$. Then

$$\sum_{j=1}^{m} [\exp(ik(x\cos\theta_j + y\sin\theta_j)) - \exp(ik(x\cos(-\theta_j) + y\sin(-\theta_j)))] = 0 \tag{2.1}$$

along the x-axis. Here the cancellation along $y = 0$ occurs in pairs.

(c) Interference of an even number of waves along a scatterer need not always occur pairwise. To see this it suffices (by (a)) to exhibit a nonlinear scatterer for four waves. Take S as the curve defined implicitly by $\sin x = 2 \sin y$. Hence the sum

$$\left[\frac{\exp(ix) - \exp(-ix)}{2i}\right] - 2\left[\frac{\exp(iy) - \exp(-iy)}{2i}\right] = 0 \tag{2.2}$$

along S. This scatterer is not a closed curve (Fig. 2). However the eight plane waves defined by putting $u = \sin 3x \sin y + \sin x \sin 3y$ into exponential notation, corresponding to $k = 10^{1/2}$, cancel along the closed oval

$$\cos^2 x + \cos^2 y = \tfrac{1}{2}, \tag{2.3}$$

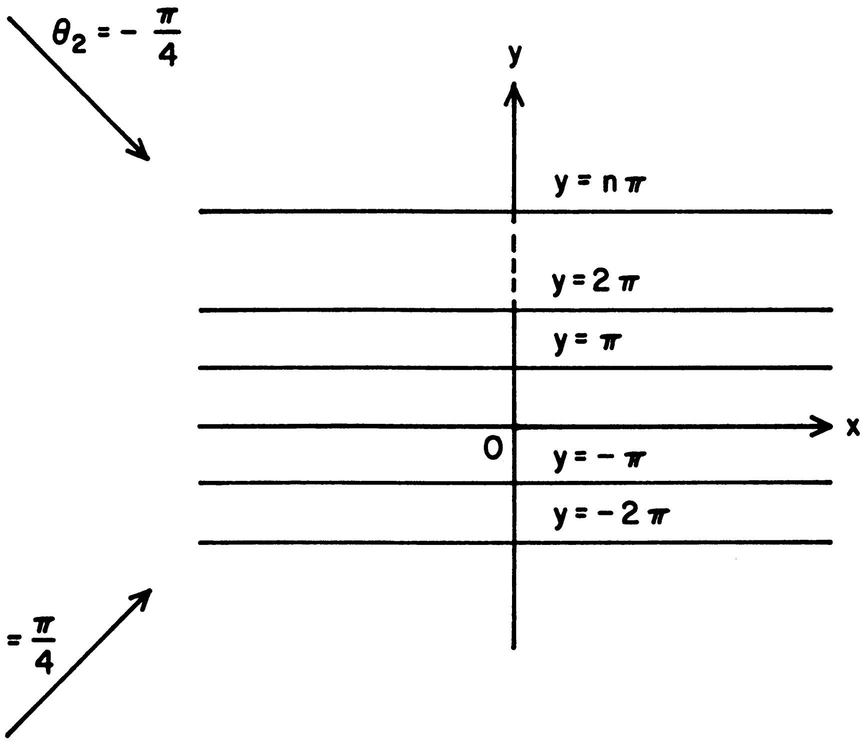

FIG. 1. The multiple scatterer $y = n\pi$, $n = 0, \pm 1, \pm 2, \cdots$, for the sum $\exp i(x+y) - \exp i(x-y) = 2i \exp(ix) \sin y$. Here $A_1 = 1$, $A_2 = -1$, $\theta_1 = \pi/4$, $\theta_2 = -\pi/4$.

defined by Rayleigh [2]. To see this we note that

$$\sin 3x \sin y + \sin x \sin 3y = 4 \sin x \sin y (\cos^2 x + \cos^2 y - \tfrac{1}{2}). \tag{2.4}$$

Here again the cancellation along S is not in pairs but requires the participation of all eight waves (Fig. 3).

The method used in (c) of obtaining scatterers implicitly by putting sums of products of sines and cosines of appropriate frequencies into exponential form is quite general. However this procedure always yields an even number of waves, which lends credence to the conjecture. We do not know of any other general method for producing scatterers and waves. This method nevertheless yields a rich supply of scatterers. For example, the sum

$$-4 \sin \pi x \sin \pi y = \exp(i\pi(x+y)) - \exp(i\pi(x-y)) - \exp(i\pi(-x+y)) + \exp(i\pi(-x-y)) \tag{2.5}$$

possesses the lattice scatterer

$$S\colon x = m, y = n;\ m, n = 0, \pm 1, \pm 2, \cdots. \tag{2.6}$$

If these four plane waves are incident on a finite portion of the grating S there will be no scattered field (Fig. 4). On each segment of S the waves cancel in pairs although the members of these pairs are different for vertical and horizontal segments. Each of the rectangles of the grating is an example of a closed (but not smooth) scatterer for four

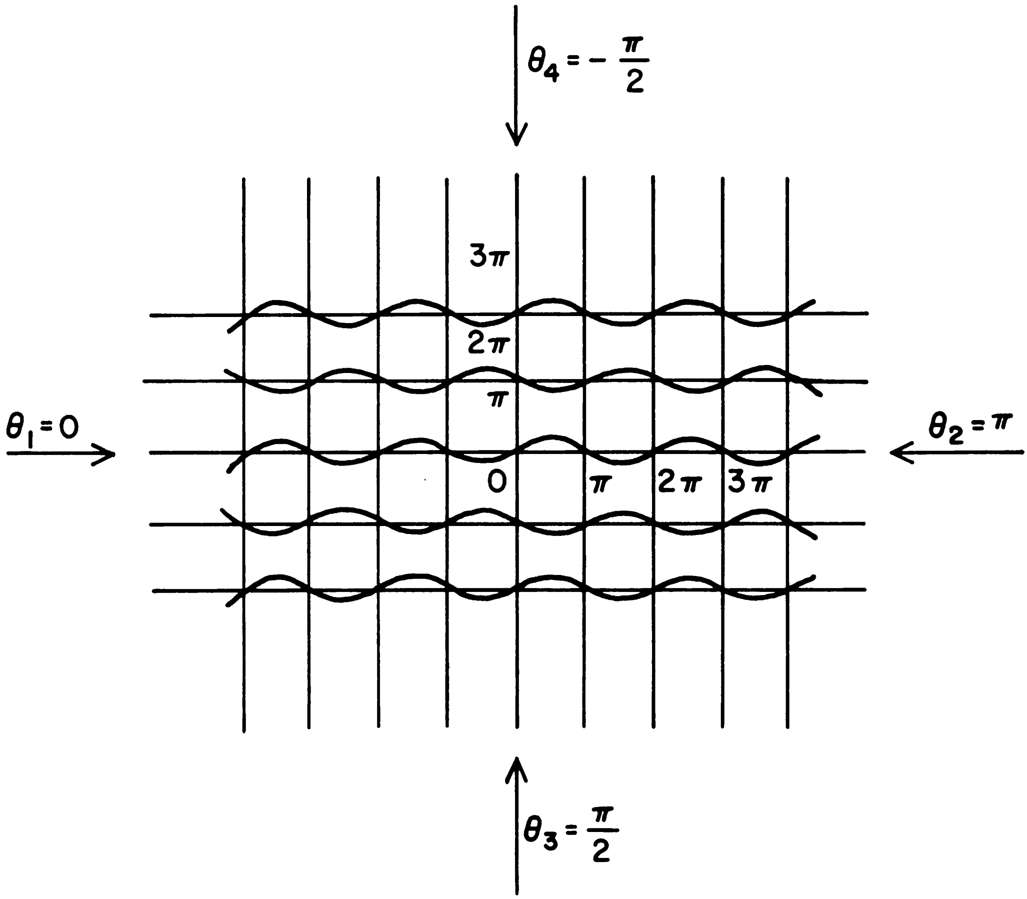

FIG. 2. $\sin x = 2 \sin y$. This is a periodic, curved scatterer for only 4 waves. $A_1 = 1/2$, $A_2 = -1/2$, $A_3 = -1$, $A_4 = 1$.

waves. We cannot come up with a smooth closed scatterer for fewer than eight waves. Many more examples are given in Rayliegh.

3. Rotation and translation. Under a rotation and translation of coordinates the sum in (1.3) goes over into a similar sum of plane waves with non-zero amplitudes, but the angles θ_j and the scatterer S will be accordingly rotated and translated. We can therefore assume for simplicity that any prospective scatterer goes through the origin with prescribed slope.

4. The case of three waves. For $n = 1$ the conjecture is trivial. Karp's original proof for $n = 3$ assumes that S is a C^2 arc. We now present a proof based only on the continuity of S. Neither of these admit any easy generalization, even to $n = 5$.

THEOREM 4.1. Three waves cannot possess a scatterer.

Proof: Suppose that along some S

$$\sum_{j=1}^{3} A_j \exp\left(ik(x \cos \theta_j + y \sin \theta_j)\right) = 0, \tag{4.1}$$

where the A_j do not vanish. It suffices to show that two of the angles must coincide. We may rewrite (4.1) in the form

$$A_1 \exp (ik[x(\cos \theta_1 - \cos \theta_3) + y(\sin \theta_1 - \sin \theta_3)]) + A_2 \exp (ik[x(\cos \theta_2 - \cos \theta_3) + y(\sin \theta_2 - \sin \theta_3)]) = -A_3 . \quad (4.2)$$

If we multiply (4.2) by its conjugate and simplify we obtain

$$x(\cos \theta_1 - \cos \theta_2) + y(\sin \theta_1 - \sin \theta_2) = \frac{1}{k} (\cos^{-1} a\text{-arg } A_1 + \arg A_2), \quad (4.3)$$

where

$$a = \frac{1}{2|A_1||A_2|} (|A_3|^2 - |A_1|^2 - |A_2|^2). \quad (4.4)$$

In a similar way we may divide (4.1) by its second term and obtain

$$x(\cos \theta_1 - \cos \theta_3) + y(\sin \theta_1 - \sin \theta_3) = \frac{1}{k} (\cos^{-1} b\text{-arg } A_1 + \arg A_3), \quad (4.5)$$

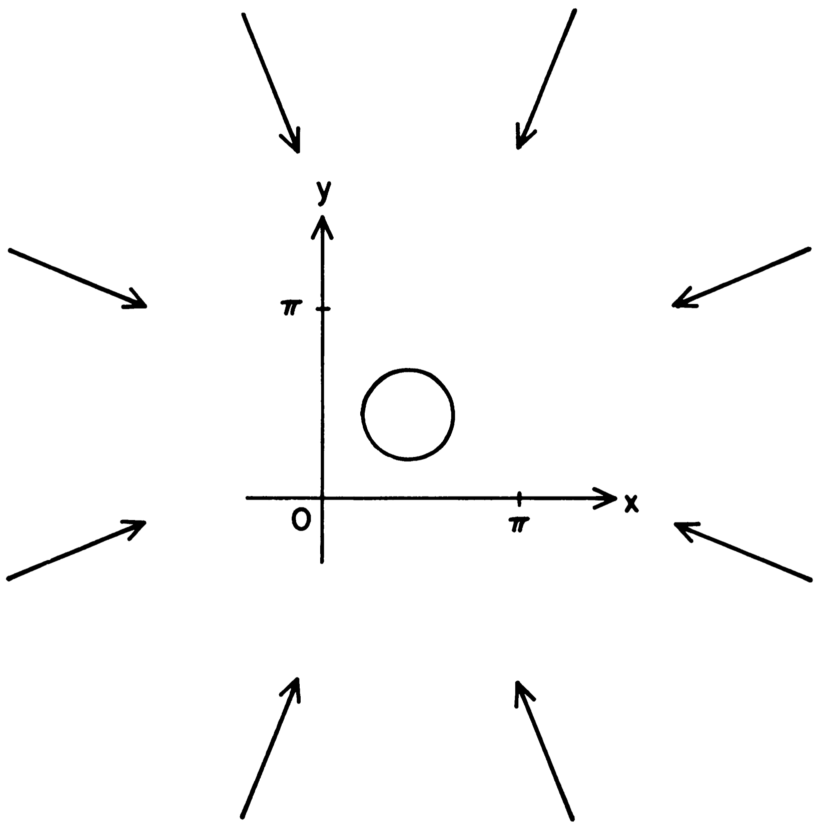

FIG. 3. Rayleigh's oval. A smooth, closed scatterer for 8 waves. The incoming rays are in the directions $(j\pi/2) \pm \theta, j = 0, 1, 2, 3$, where $\cos \theta = 3/\sqrt{10}$, $\sin \theta = 1/\sqrt{10}$. There is no scattered field from (any segment of) the oval.

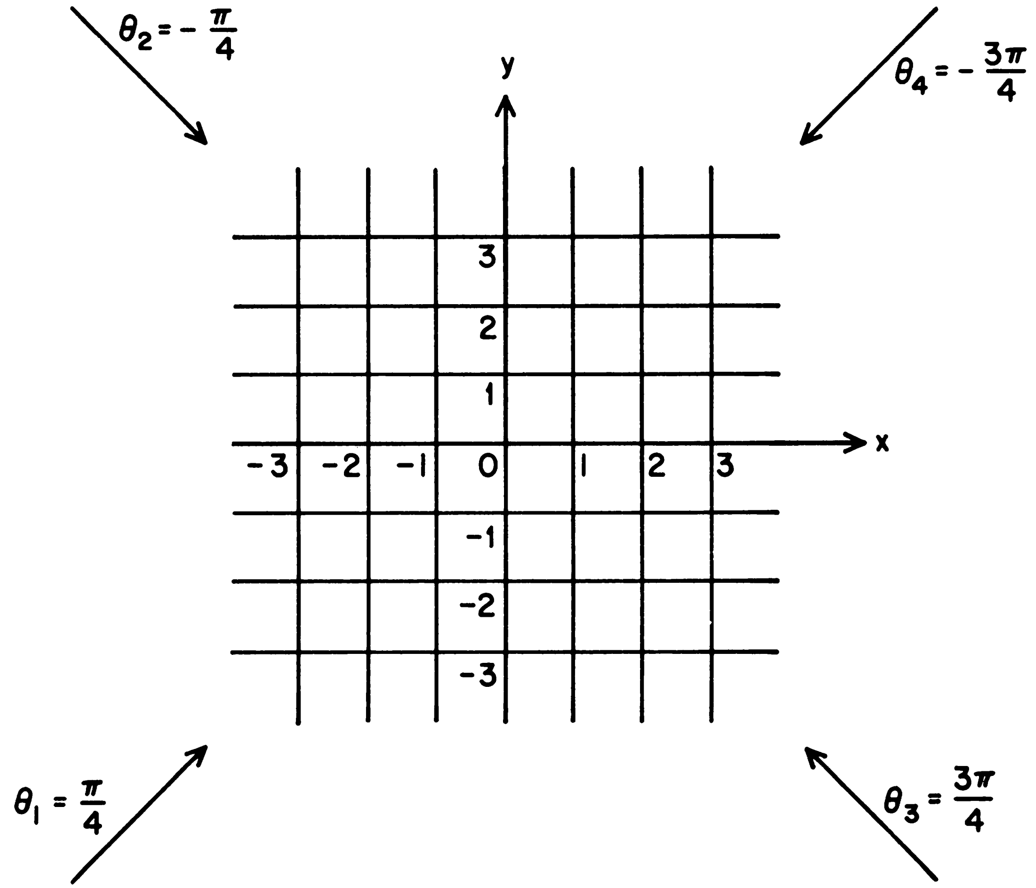

FIG. 4. The grating $x = m, y = n; m, n = 0, \pm 1, \pm 2, \cdots$, defined by $\sin \pi x \sin \pi y = 0$. Here $A_1 = 1, A_2 = -1, A_3 = -1, A_4 = 1, \theta_1 = \pi/4, \theta_2 = -\pi/4, \theta_3 = 3\pi/4, \theta_4 = -3\pi/4$. Any finite portion is a scatterer for which the scattered field is zero.

where

$$b = \frac{1}{2|A_1||A_3|}(|A_2|^2 - |A_1|^2 - |A_3|^2). \tag{4.6}$$

Since every point on S is a solution of the linear system (4.3), (4.5), the determinant

$$\begin{aligned}(\cos \theta_1 - \cos \theta_2)(\sin \theta_1 - \sin \theta_3) - (\sin \theta_1 - \sin \theta_2)(\cos \theta_1 - \cos \theta_3) \\ = \sin (\theta_2 - \theta_1) + \sin (\theta_3 - \theta_2) + \sin (\theta_1 - \theta_3)\end{aligned} \tag{4.7}$$

of the system must vanish. It is easy to see that (4.7) then implies that at least two of the angles are equal. (The continuity of S was used in the assumption that the multiple-valued right-hand sides of (4.3) and (4.5) remain constant on S).

Remark: This proof can be done geometrically by representing (4.1) by vectors in the complex plane. They form a triangle with a vertex at the origin which varies as (x, y) moves along S. The triangle must move as a rigid body (i.e., rotate) since the lengths $|A_1|$, $|A_2|$, $|A_3|$ of its sides are fixed. The analytical expression of the fact that the exterior angles of such a triangle remain constant is essentially (4.3) and (4.5). Unfortunately a pentagon ($n = 5$) need not move as a rigid body when its sides are of fixed lengths. On the other hand, our attempt to find a possible scatterer for five waves by seeing just how such a pentagon could distort was unsuccessful.

5. Linear scatterers. The following theorem shows that the conjecture holds for linear scatterers.

THEOREM 5.1. An odd number of plane waves cannot possess a linear scatterer.

Proof: Suppose that n waves possess the linear scatterer S, which we may assume is the x-axis. Then (1.3) becomes

$$\sum_{j=1}^{n} A_j \exp(ikx \cos \theta_j) = 0 \tag{5.1}$$

for all x. Since the angles are distinct, no three of the $\cos \theta_j$ are equal. Combining terms with equal $\cos \theta_j$, we obtain

$$\sum_{p=1}^{m} B_p \exp(ikx \cos \theta_{j_p}) = 0, \tag{5.2}$$

where the $\cos \theta_{j_p}$ are distinct and each B_p is either an A_j or the sum of two A_js. Since the exponentials in (5.2) are linearly independent the coefficients vanish. Since no A_j vanishes, each B_p is the sum of two A_js and $n = 2m$ is even.

Remark: This argument also shows that if an even number of plane waves interfere along a line the waves must cancel in pairs.

6. Almost linear scatterers. The results of the previous section can be generalized to the case of a curved scatterer which contains a point at which it is appropriately flat. We will say that the *smooth* curve S given by $y = g(x)$, $a \leq x \leq b$ is linear to order $k \geq 1$ at $(x_0, y_0) \in S$ when

$$y - y_0 - g'(x_0)(x - x_0) = o((x - x_0)^k) \text{ as } x \to x_0, \tag{6.1}$$

i.e., the tangent line at (x_0, y_0) approximates the curve faster than $(x - x_0)^k$. The next theorem shows that the greater the value of k, the greater is the odd number of waves required for interference along S; in the limiting case where S is a straight line it reduces to Theorem 5.1. We will need the following lemmas, the first of which is established by simple estimates.

LEMMA 6.1. If r is any non-negative integer then

$$exp\ o(x^r) = 1 + o(x^r) \text{ as } x \to 0. \tag{6.2}$$

LEMMA 6.2. Let

$$\sum_{j=1}^{r+1} C_j \exp(i\lambda_j x) = o(x^r) \quad \text{as} \quad x \to 0, \tag{6.3}$$

where r is a non-negative integer, the λ_j are real and distinct, and the C_j are complex. Then $C_1 = C_2 = \cdots = C_{r+1} = 0$.

Proof: Repeated application of L'Hospital's rule to

$$\lim_{x \to 0} x^p \frac{\sum_{j=1}^{r+1} C_j \exp(i\lambda_j x)}{x^r} = 0, p = r, r-1, \cdots, 0 \tag{6.4}$$

leads to a linear homogeneous system of $r + 1$ equations for the C_j with non-vanishing (Vandermonde) determinant.

THEOREM 6.1. If n waves interfere along a smooth scatterer S containing a point at which it is linear to order k, and n is odd, then $n > k + 1$.

Proof: It is easily established that the order of linearity of a point is invariant under rotation and translation. Hence we may assume S is given by

$$y = o(x^k) \tag{6.5}$$

near the origin. The interference condition is then

$$\sum_{j=1}^{n} A_j \exp\left[ik(x \cos\theta_j + o(x^k)\sin\theta_j)\right] = 0 \tag{6.6}$$

near $x = 0$. Applying Lemma 6.1, this becomes

$$\sum_{j=1}^{n} A_j \exp(ikx \cos\theta_j)(1 + o(x^k)) = 0, \tag{6.7}$$

or

$$\sum_{j=1}^{n} A_j \exp(ikx\cos\theta_j) = -\sum_{j=1}^{n} o(x^k)A_j \exp(ikx\cos\theta_j) = o(x^k). \tag{6.8}$$

As in (5.2) we may combine terms with equal $\cos\theta_j$ on the left of (6.8) and obtain

$$\sum_{p=1}^{r+1} B_p \exp(ikx\cos\theta_{j_p}) = o(x^k) \quad \text{as} \quad x \to 0 \tag{6.9}$$

with distinct $\cos\theta_{j_p}$ and $r + 1 \leq n$. Suppose $n \leq k + 1$. Then we would have $r \leq k$ and hence

$$\sum_{p=1}^{r+1} B_p \exp(ikx\cos\theta_{j_p}) = o(x^r) \quad \text{as} \quad x \to 0, \tag{6.10}$$

so that by Lemma 6.2 the B_p would vanish, implying, as before, the evenness of n.

7. Other attempts. One might attempt to use Theorem 6.1 to prove the conjecture by employing a change of variables to flatten out any proposed scatterer at one of its points. This approach appears futile since such changes of variables do not seem to carry plane waves into plane waves.

Any scatterer is a node of a sum of trigonometric functions and is therefore an analytic curve. Accordingly one might try substituting a power series for $y = g(x)$ into the interference condition (1.3) and attempt to solve for the unknown *real* coefficients. We have done this but were not able to ascertain whether or not the resulting recurrence relations had real solutions. The existence of such solutions (with n odd) would disprove the conjecture.

Any approach to a proof of the conjecture will have to distinguish between even and odd n and not imply pairwise cancellation for even n. A useful fact in this connection is the following. Let

$$\sum_{j=1}^{n} A_j \cos^{i-1}\theta_j = 0, \qquad i = 1, 2, \cdots, n, \tag{7.1}$$

where no A_j vanishes. If n is odd then at least two angles are equal.

We have not investigated the three-dimensional case or the case of more general boundary conditions.

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[3] Leo M. Levine, *A uniqueness theorem for the reduced wave equation*, Comm. Pure Appl. Math. **17**, 147–175 (1964) (see section 7).