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OF  
APPLIED MATHEMATICS

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VOLUME XXXV

APRIL • 1977

NUMBER 1

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# QUARTERLY OF APPLIED MATHEMATICS

The QUARTERLY prints original papers in applied mathematics which have an intimate connection with applications. It is expected that each paper will be of a high scientific standard; that the presentation will be of such character that the paper can be easily read by those to whom it would be of interest; and that the mathematical argument, judged by the standard of the field of application, will be of an advanced character.

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Second-class postage paid at Providence, Rhode Island, and at Richmond, Virginia

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1. TITLE OF PUBLICATION <b>Quarterly of Applied Mathematics</b>		2. DATE OF FILING <b>October 1, 1976</b>	
3. FREQUENCY OF ISSUE <b>Quarterly</b>		A. NO. OF ISSUES PUBLISHED ANNUALLY <b>4</b>	B. ANNUAL SUBSCRIPTION PRICE <b>\$25.00(1976)</b>
4. LOCATION OF KNOWN OFFICE OF PUBLICATION (Street, City, County, State and ZIP Code) (Not printers) <b>Brown University, Providence, Rhode Island 02912</b>			
5. LOCATION OF THE HEADQUARTERS OR GENERAL BUSINESS OFFICES OF THE PUBLISHERS (Not printers) <b>Brown University, Providence, Rhode Island 02912</b>			
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The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

**Titles:** The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

**Mathematical Work:** As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ('), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent  $\frac{1}{2}$  rather than with the sign  $\sqrt{\quad}$ .

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp [(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos (\pi x / 2 b)}{\cos (\pi a / 2 b)} \text{ is preferable to } \frac{\cos \frac{\pi x}{2 b}}{\cos \frac{\pi a}{2 b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u \, du \text{ is preferable to } \int \frac{\sin u}{u} \, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t(a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

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The following examples show the desired arrangements: (*for books*—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354–372(1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten*.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

**Footnotes:** As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

**Abbreviations:** Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq. (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.



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ALAN L. MACKAY	



## —BOOK REVIEW SECTION—

*Generalized inverses and applications.* Edited by M. Z. Nashed. Academic Press, New York, 1976. xiv + 1054 pp. \$34.00.

This book is the proceedings of an advanced seminar on generalized inverses and applications held at the Mathematics Research Center, University of Wisconsin, Madison, Wisconsin, October, 1973. It contains a historical preface, thirteen articles, and an annotated bibliography containing 1775 references. If the authors of this bibliography (Nashed and Rall) had known about an additional paper on generalized inverses by the reviewer and lying incomplete on his desk, they obviously would not have had to list Jefferson's Declaration of Independence in order to bring the bibliography up to 1776 items in honor of the U.S.A. Bicentennial.

The articles are divided into four main areas: Theory of generalized inverses: M. Z. Nashed and G. F. Votruba, R. E. Kalman, A. Ben-Israel and T. N. E. Greville; Generalized inverses in analysis: L. B. Rall, M. R. Hestenes, M. Z. Nashed; Computational methods and approximation theory: B. Noble, G. H. Golub and V. Pereyra, M. Z. Nashed; Applications: F. J. Beutler and W. L. Root, A. Ben-Israel, Arthur Albert, J. S. Chipman. One of these articles (Chipman on estimation and aggregation in econometrics) is itself of book length. This is an important addition to the MRC's fine list of publications.

What is the theory of generalized inverses? In a word, and considering the matrix case, it is an attempt to deal, symbolically and otherwise, with the linear system  $Ax = b$  when  $A$  is not square or when  $A$  is square and is not invertible. It is clear how this simple concern can then diffuse throughout the whole of linear mathematics and its applications.

A personal note: I first heard about generalized inverses in the mid-50's in a talk presented by T. N. E. Greville at the National Bureau of Standards. I found the subject appealing and have maintained a kind of amateur's interest in it over the years. I also thought it would be useful as a unifying tool. But, ye Gods, 1775 references in 20 years! How can an amateur keep up with such activity? I have read overviews (in this field and elsewhere) and it just doesn't work. To mix a metaphor: I find myself drowning in undigested material. Help!

One final observation: in view of one-thousand-page unifications, it would appear that there are no really unifying themes in mathematics.

P. J. DAVIS (*Providence*)

*Pattern synthesis: lectures in pattern theory.* Volume 1. By Ulf Granander. Springer-Verlag, New York, Heidelberg, Berlin, 1976. 510 pp. \$14.50.

The problem of growth and form is one which has always been with us. Each generation of scientists makes its own attempt to explain the baffling phenomena of morphogenesis which characterize our world, particularly the living world. Indeed, organizing one's sense impressions and seeing regular patterns in them is one of the main characteristics of intelligent life. We must jump from the attempt of Democritus to explain everything in terms of a few kinds of atoms to D'Arcy Thompson's classic *On Growth and Form* (1916).

In essence, D'Arcy Thompson pointed out that many of the patterns to be observed in nature could be described in simple mathematical terms. That is, he carried the Galilean mathematization of the physical world into the biological, building on the continuous algebraic descriptions developed by the great mathematicians of the Enlightenment. Although the particulate nature of matter and the quantization of energy were then known, D'Arcy Thompson did not penetrate below the appearances to the next substrate of mechanism. It is only in our own time, first in chemistry, and then, since the



double helix was conclusively demonstrated, in biology, that molecular mechanisms have been taken seriously.

Today's approaches are dominated by the concept algorithmic. In many systems, particularly those of living matter, we see the segregation of information into special information-carrying structures (still made of the same atoms) which serve to organize the development of the remaining material and to produce patterns.

Professor Grenander's project, then, of which this is only the first volume, is nothing less than to describe all types of patterns. Applying Chaitin's criterion (*Scientific American*, May, 1975, pp. 47-52), and earlier work by Martin Löf, we can define a pattern as being a structure which can be described in terms of generators with fewer bits (of information) than are necessary to describe the whole structure directly as it stands. That is, some rule for compressing the description is discernible. H. A. Fatmi and R. W. Young (*Nature*, 228, 97, 3 Oct. 1970) have defined intelligence as "that faculty of mind, by which order is perceived in a situation previously considered disordered".

Grenander, then, examines a very wide variety of patterns in space and time and uses his insight and skill to show how they may be generated by relatively simple instructions. In this first volume the generation of patterns is examined. Extending the theme, the distortion of images by various transfer functions is also included. The patterns considered range from crystal structures to PERT (schemes for the evaluation of the order in which to do various jobs in a construction project).

It must be said that for a non-mathematician (such as the reviewer) much of the treatment seems somewhat complicated. The principle should be that a problem is translated into mathematical terms ("projected into a mathematical space"), it is then manipulated in the exact mill of mathematics, and the product is then projected back again into "problem space". The resultant is then compared with the original, with the hope that some illumination will result. That is, it is hoped that the information necessarily lost in the two operations of projection will prove to have been insignificant, and that the significant features of the situation thus have been correctly isolated.

The theory will be of value if it shows us something which we have not seen before. The test is user acceptance. All the patterns discussed are being attacked by specialists who probably will not take any notice of overall treatment unless either the mathematical formulation enables some outstanding problem to be solved or juxtaposition shows that their problem has already been solved in an isomorphous case.

The study involves a taxonomy of patterns. The most significant taxonomy is one where the objects themselves are actually interrelated in an evolutionary way originating from each other (as, for example, do insects). If the objects are not so related (as, for example, the crystal structures of minerals), taxonomy is of only bureaucratic rather than of scientific significance.

Although crystal patterns are well understood and readily organized, the patterns of speech still elude us, although progress is being made. As in amino-acid sequences, the invocation of statistical elements is a confession of temporary ignorance of the detailed mechanisms.

Grenander's project is to be welcomed as an interesting and fundamental effort, although at present it appears rather widely cast and eclectic. We must look forward to the complete study where results are pulled together. Later volumes will be devoted to the analysis of patterns, to the ways in which they are processed and to the formulation of a general theory.

ALAN L. MACKAY (*London*)