APRIL, 1977

CONTINUOUS DATA-DEPENDENT RESULTS FOR A GENERAL THEORY OF HEAT CONDUCTION IN BOUNDED AND UNBOUNDED DOMAINS*

By

C. E. BEEVERS

Heriot-Watt University, Edinburgh

1. Introduction. The mathematical and physical shortcomings of Fourier's law of heat conduction are well known. However, with the development of history-dependent theories of heat conduction, some of the unsatisfactory features of the classical theory are removed. In particular, the prediction of finite thermal pulse propagation is possible within these general theories. Notable amongst these general theories are those proposed by Coleman and Gurtin [1] and Gurtin and Pipkin [2]. The linear theory of [1] is constructed by Nunziato [3]. It differs from the linear theory in [2] since the theory of Coleman and Gurtin allows the heat flux vector at time t, $\mathbf{q}(t)$, to depend not only on the history of the temperature gradient but also on the present value of the temperature gradient.

In this article we are concerned with the linear theory of [1]. So in the next section we set out the anisotropic form of the theory established in [3]. After further necessary preliminaries in Sec. 2 we state the conditions in Theorem 1 of Section 3 which ensure that solutions of the equations governing heat flow depend continuously on the history before $t = 0$, the initial value of the temperature, and the heat supply r per unit volume and time (which are collectively referred to as the 'data' in the sequel). This result is valid for any bounded domain D of Euclidean three-space $\mathbf{R}^3$ which possesses a boundary Σ smooth enough to permit applications of the divergence theorem. We complete Sec. 3 with a corollary to Theorem 1 in the form of a uniqueness theorem which generalizes results in [3,4]. In the final section we extend the scope of Theorem 1 so that it becomes valid on the exterior $\mathcal{E}$ of a bounded domain in $\mathbf{R}^3$ with

$$\mathcal{E} = \mathbf{R}^3 \backslash \bar{D},$$

where $\bar{D}$ is the closure of D. This result is embodied in Theorem 3. A corollary of Theorem 3 provides a new uniqueness result for the linearized version of the theory in [1] which applies to some unbounded regions.

2. Equations, definitions and notation. In order to put this present work in the context of existing results, we begin by stating the mixed history boundary-value problem for the linear theory in [1]. We observe that the theory analyzed here is the anisotropic form of the theory derived by Nunziato [3]. We employ the usual index notation with summation over repeated indices, and Latin lower-case indices range over the values 1, 2, 3. Let x_i be the components of a typical place $\mathbf{x} \in V$ where for now V denotes

* Received January 23, 1976.

either D or $\mathcal{E}$. The time is restricted to the open interval $(-\infty, T)$ where T is finite.

For anisotropic, inhomogeneous rigid heat conductors the internal energy e and the heat flux vector satisfy the equations:

$$\dot{e} = r - q_{i,i}\ ; \tag{2.1}$$

$$e(\mathbf{x}, t) = c(\mathbf{x})\theta(\mathbf{x}, t) + \int_{-\infty}^{t} \beta(\mathbf{x}, t - \tau)\theta(\mathbf{x}, \tau)\, d\tau; \tag{2.2}$$

and

$$q_i(\mathbf{x}, t) = -\kappa_{ij}(\mathbf{x})g_j(\mathbf{x}, t) - \int_{-\infty}^{t} \alpha_{ij}(\mathbf{x}, t - \tau)g_j(\mathbf{x}, \tau)\, d\tau, \tag{2.3}$$

where θ is the temperature deviation from some homogeneous reference temperature, $\mathbf{g} = \nabla\theta$ is the temperature gradient, c is the specific heat, $\boldsymbol{\kappa}$ is the instantaneous thermal conductivity tensor and $\boldsymbol{\alpha}$ and β are further material constants. A superposed dot denotes the time derivative and a comma followed by the index i represents differentiation with respect to x_i. We observe that if $\boldsymbol{\kappa}$ is identically zero then the heat flux vector is independent of the present value of the temperature gradient and in this case the linear theories of [1, 2] coincide.

Upon combining Eqs. (2.1)–(2.3) we obtain the single equation governing small temperature deviations:

$$c\dot{\theta}(t) + \beta(0)\theta(t) + \int_0^t \dot{\beta}(t - \tau)\theta(\tau)\, d\tau = Q(t) + \left\{\kappa_{ij}g_j(t) + \int_{-\infty}^{t} \alpha_{ij}(t - \tau)g_j(\tau)\, d\tau\right\}_{,i}, \tag{2.4}$$

where

$$Q(t) = r(t) - \int_{-\infty}^{0} \dot{\beta}(t - \tau)\theta(\tau)\, d\tau. \tag{2.5}$$

Eq. (2.4) is valid throughout $V \times (0, T)$ and we have omitted explicit reference to $\mathbf{x}$ for convenience. The mixed thermal problem is completed by specifying the boundary conditions:

$$\mathbf{q}(\mathbf{x}, t)\cdot\mathbf{n} = q^*(\mathbf{x}, t), \qquad \Sigma_1 \times (0, T);\ \theta(\mathbf{x}, t) = \theta^*(\mathbf{x}, t),\ \Sigma_2 \times (0, T) \tag{2.6}$$

where $\mathbf{n}$ is the unit outward normal on Σ_1 and $\Sigma_1 \cup \Sigma_2 = \Sigma$, together with the history

$$\theta(\mathbf{x}, t) = \theta_h(\mathbf{x}, t), \qquad \bar{V} \times (-\infty, 0]. \tag{2.7}$$

We will refer to Eqs. (2.4)–(2.7) as the model $\mathfrak{M}$. We restrict $\mathfrak{M}$ by supposing that

$$c \geq c_m > 0, \qquad \kappa_{ij}\eta_i\eta_j \geq \kappa\eta_i\eta_i > 0; \tag{2.8}$$

and

$$A = \max_{\bar{V}\times[0,T]} \{|\alpha_{ij}|, |\alpha_{li}\alpha_{lj}|, |\kappa_{li}\kappa_{lj}|, |\beta(0)|, |\dot{\beta}|\}. \tag{2.9}$$

The model $\mathfrak{M}$ has the desirable feature of predicting finite thermal wave speeds. This property has proved useful in theoretical calculations on experimental observations in some materials at low temperatures (see for example Nunziato [3]). With $\boldsymbol{\kappa} = \mathbf{0}$ the wave propagation property has also been employed by Finn and Wheeler [5] to investigate uniqueness for an isotropic, homogeneous version of $\mathfrak{M}$. The proof in [5] is valid for unbounded domains and follows the method outlined in the comprehensive article on linear elastodynamics by Wheeler and Sternberg [6]. More recently Herrmann and Nachlinger [7, 8] have utilized the wave propagation property to establish some uniqueness results for a nonlinear version of $\mathfrak{M}$. However, all the theorems in [7, 8] for unbounded regions require κ to be identically zero. In [8] the authors do prove a result for a nonlinear form of $\mathfrak{M}$ with conditions equivalent to (2.8) above, but their result is restricted to bounded domains D only.

In this present approach we introduce a combination of two methods due to Protter and Graffi. This combination is different in concept from that adopted in [5–8] and it leads to new results for $\mathfrak{M}$ given by Theorems 3 and 4 of Sec. 4 below. The Protter method is essentially an energy argument and has been used by the present author to determine some uniqueness and stability results for linearly viscoelastic materials contained in a bounded domain D (for further details of this method the reader is referred to [9]). Strictly speaking it is an adaptation of the Protter method which is employed here in Sec. 3 to prove Theorems 1 and 2. Then in the final section the Graffi method is applied to extend the scope of Theorems 1 and 2 into the exterior $\mathcal{E}$ of a bounded domain. Graffi [10] employs his method to construct uniqueness results for solutions of the Navier-Stokes equations. The Graffi method also appears in [11–13] where analytic continuation, uniqueness and continuous dependence results are presented. The combination of the Protter and Graffi methods is quite novel and provides an alternative approach to that described in [5–8].

We fix our origin of coordinates within D. Let $S(R)$ be the sphere of radius R centred at the origin and find the smallest radius $R = R_m$ such that $S(R_m) \supseteq \bar{D}$. Then, for $R \in [R_m, \infty)$ we put

$$\Omega = \Omega(R) = S(R)\backslash\bar{D} \tag{2.10}$$

and we take $\sigma = \sigma(R)$ as the surface of $S(R)$.

Next, we define the following measures over V where V now denotes either D or Ω:

$$h(t) = \frac{1}{2}\int_V c\theta^2(t)\,dx; \qquad g(t) = \int_V g_i(t)g_i(t)\,dx, \tag{2.11}$$

where dx is the volume element. We note that when $V = \Omega$ both the above measures are also functions of R in general and so in Sec. 4 we will write $h(t) = h(t, R)$ and $g(t) = g(t, R)$ to emphasize this dependence.

Moreover, we define

$$H(t) = \int_0^t h(\tau)\exp(-\mu\tau)\,d\tau; \qquad G(t) = \int_0^t (t-\tau)g(\tau)\exp(-\mu\tau)\,d\tau, \tag{2.12}$$

and

$$E(t) = H(t) + \tfrac{1}{4}\kappa G(t), \tag{2.13}$$

where μ is a constant yet to be determined.

Finally, we observe that in seeking continuous dependence results with the dependence on the "data" it is sufficient to take

$$\mathrm{q}^* = \theta^* = 0 \tag{2.14}$$

in Eqs. (2.6). We will make the necessary assumptions on the initial value of the temperature, the history of the temperature before $t = 0$ and the heat supply at appropriate stages in the sequel. For complete generality we could include changes in q^* and θ^*, but this is not attempted here.

3. Some results on a bounded domain. On multiplying Eq. (2.4) at $t = \tau$ by $\theta(\tau)$ and integrating over V we obtain

$$\begin{aligned} \dot{h}(\tau) + \kappa g(\tau) \leq \int_V \{Q(\tau)\theta(\tau) + \alpha_k(\tau)g_k(\tau) - \beta(0)\theta^2(\tau)\}\, dx \\ - \int_V \int_0^\tau \{\dot{\beta}(\tau - s)\theta(\tau)\theta(s) + \alpha_{ij}(\tau - s)g_i(\tau)g_j(s)\}\, ds\, dx \\ + \int_\sigma \left\{\kappa_{ij}g_j(\tau) - \alpha_i(\tau) + \int_0^\tau \alpha_{ij}(\tau - s)g_j(s)\, ds\right\} x_i\theta(\tau)\, \frac{d\sigma}{R}, \end{aligned} \tag{3.1}$$

where

$$\alpha_i(t) = -\int_{-\infty}^0 \alpha_{ij}(t - \tau)g_j(\tau)\, d\tau,$$

$d\sigma$ is the surface element on σ and we have applied the divergence theorem, conditions $(2.8)_2$ and (2.14) together with definitions (2.11). We notice that if $V = D$ then the surface integral does not appear in inequality (3.1). It is from this fundamental inequality that all the results in this paper are derived.

Now, we multiply inequality (3.1) by $\exp(-\mu\tau)$ for some positive μ and on integration over $[0, t]$ it yields:

$$\begin{aligned} & h(t) \exp(-\mu t) + \mu H(t) \\ & + \kappa \int_0^t g(\tau) \exp(-\mu\tau)\, d\tau \leq h(0) + \int_0^t \int_V \left\{Q\theta + \alpha_k g_k\right\}(\tau) \exp(-\mu\tau)\, dx\, d\tau \\ & - \int_0^t \int_V \left\{\beta(0)\theta(\tau) + \int_0^\tau \dot{\beta}(\tau - s)\theta(s)\, ds\right\}\theta(\tau) \exp(-\mu\tau)\, dx\, d\tau \\ & - \int_0^t \int_V \int_0^\tau \alpha_{ij}(\tau - s)g_i(\tau)g_j(s) \exp(-\mu\tau)\, ds\, dx\, d\tau \\ & + \int_0^t \int_\sigma \left\{\kappa_{ij}g_j(\tau) - \alpha_i(\tau) + \int_0^\tau \alpha_{ij}(\tau - s)g_j(s)\, ds\right\} x_i\theta(\tau) \exp(-\mu\tau)\, \frac{d\sigma\, d\tau}{R}. \end{aligned} \tag{3.2}$$

We now invoke two, one weighted, arithmetic-geometric mean inequalities to show that

$$-\int_0^t \int_V \left\{\beta(0)\theta(\tau) + \int_0^\tau \dot{\beta}(\tau - s)\theta(s)\, ds\right\}\theta(\tau) \exp(-\mu\tau)\, dx\, d\tau \leq \frac{2A}{c_m}(1 + T)H(t), \tag{3.3}$$

and

$$\begin{aligned} & -\int_0^t \int_V \int_0^\tau \alpha_{ij}(\tau - s)g_i(\tau)g_j(s) \exp(-\mu\tau)\, ds\, dx\, d\tau \\ & \qquad \leq \frac{1}{4}\left(\kappa + \frac{36A^2T}{\kappa\mu}\right) \int_0^t g(\tau) \exp(-\mu\tau)\, d\tau. \end{aligned} \tag{3.4}$$

In the remainder of this section we restrict our attention to bounded domains only and we take $V = D$. We say that $\theta(t)$ is a classical solution of $\mathfrak{M}$ if:

(i) it is continuously differentiable with respect to time and twice continuously differentiable with respect to space;

(ii) $\theta(t)$ satisfies Eqs. (2.4)–(2.7) and (2.14) for given "data".

We assume that such solutions exist and in this section we suppose that

$$\max_{t\in[0,T]}\left\{h(0),\ \int_D Q^2(t)\,dx,\ \int_D \alpha_k(t)\alpha_k(t)\,dx\right\} = \epsilon, \qquad 0 < \epsilon < 1. \tag{3.5}$$

We can now state and prove Theorem 1.

Theorem 1. If $\theta(t)$ is a classical solution of the model $\mathfrak{M}$ subject to conditions (2.8) and (2.9) and provided the measure ϵ is sufficiently small, then the solution, as measured by $h(t)$, depends Hölder-continuously on its "data" in compact subsets of $[0, T)$.

Proof. Two arithmetic-geometric mean inequalities are used to show that

$$\int_0^t \int_D \{Q\theta + \alpha_k g_k\}(\tau)\exp(-\mu\tau)\,dx\,d\tau \leq H(t) + \tfrac{1}{4}\kappa\int_0^t g(\tau)\exp(-\mu\tau)\,d\tau + \left(\frac{1}{2c_m} + \frac{1}{\kappa}\right)\frac{\epsilon}{\mu}. \tag{3.6}$$

Then, on combining (3.2)–(3.6), we deduce that

$$h(t)\exp(-\mu t) + \left\{\mu - \left[1 + \frac{2A}{c_m}(1+T)\right]\right\}H(t) + \frac{1}{2}\left(\kappa - \frac{18A^2T}{\kappa\mu}\right)\int_0^t g(\tau)\exp(-\mu\tau)\,d\tau \leq \epsilon\left\{1 + \frac{1}{\mu}\left(\frac{1}{2c_m} + \frac{1}{\kappa}\right)\right\}. \tag{3.7}$$

So, if we choose

$$\mu = \max\left\{1 + \frac{2A}{c_m}(1+T),\ \frac{18A^2T}{\kappa^2}\right\}, \tag{3.8}$$

then inequality (3.7) simplifies to give

$$h(t) \leq \epsilon\left\{1 + \frac{1}{\mu}\left(\frac{1}{2c_m} + \frac{1}{\kappa}\right)\right\}\exp(\mu t), \qquad t \in [0, T). \tag{3.9}$$

Further, for $0 < \nu < 1$, we fix μ by putting

$$\mu = -\frac{1-\nu}{T}\log\epsilon. \tag{3.10}$$

Thus, if ϵ is sufficiently small then definition (3.10) ensures that μ is large enough to justify Eq. (3.8). Hence, (3.9) and (3.10) reduce to

$$h(t) \leq \left\{1 + \frac{1}{\mu}\left(\frac{1}{2c_m} + \frac{1}{\kappa}\right)\right\}\epsilon^{\nu}, \qquad t \in [0, T). \tag{3.11}$$

This result implies Hölder continuity (see [9] for further details of and reference to Hölder continuity) and the theorem is proved.

We also have the simple corollary:

Theorem 2. (Uniqueness Theorem). If $\theta(t)$ is a classical solution of the model $\mathfrak{M}$ with conditions (2.8), (2.9) and zero "data" then $\theta(t)$ vanishes identically for all $t \in [0, T)$.

Proof. From inequalities $(2.8)_1$ and (3.11) with zero "data" we have

$$\tfrac{1}{2}c_m \int_D \theta^2(t)\, dx \le h(t) \le 0, \qquad t \in [0, T). \tag{3.12}$$

We therefore conclude that $\theta(t) = 0$ for all $t \in [0, T)$.

4. Some theorems for an exterior domain. Throughout this section we have $V = \Omega$ and by means of the Graffi method we extend the scope of Theorems 1 and 2. We say that $\theta(t)$ is a classical solution of $\mathfrak{M}$ if in addition to (i) and (ii) of the previous section we suppose that (iii) the measure $E(t) = E(t, R)$ satisfies the condition

$$E(t, R) \le C \exp\left(\tfrac{1}{2}pR\right), \tag{4.1}$$

where C is a positive constant and $0 < p < p_1$ with

$$p_1 = \left\{4\left(\frac{A}{\kappa c_m}\right)^{1/2} \max(1, T)\right\}^{-1}. \tag{4.2}$$

We again presume that such solutions do exist.

Once more we allow changes in the initial value of the temperature, its history before $t = 0$ and the heat supply. Thus, we generalize condition (3.5) and assume that

$$\begin{aligned} \max_{[R_m,\infty)\times[0,T]} \left\{h(0, R), \int_\Omega Q^2(t)\, dx, \int_\Omega \alpha_k(t)\alpha_k(t)\, dx\right\} &= Q\,\epsilon R^3 \\ \max_{[R_m,\infty)\times[0,T]} \left\{\int_\sigma \alpha_k(t)\alpha_k(t)\, d\sigma\right\} &= Q\,\epsilon R^2 \end{aligned} \tag{4.3}$$

with Q another positive constant and $0 < \epsilon \ll 1$.

Inequalities (3.2)–(3.4) with $V = \Omega$ combine to produce

$$\begin{aligned} h(t, R) \exp(-\mu t) &+ \left\{\mu - \frac{2A}{c_m}(1 + T)\right\}H(t, R) + \frac{1}{4}\left(3\kappa - \frac{36A^2T}{\kappa\mu}\right) \\ &\cdot \int_0^t g(\tau, R) \exp(-\mu\tau)\, d\tau \le h(0, R) + \int_0^t \int_\Omega \{Q\theta + \alpha_k g_k\}(\tau) \exp(-\mu\tau)\, dx\, d\tau \\ &+ \int_0^t \int_\sigma \left\{\kappa_{ij} g_j(\tau) - \alpha_i(\tau) + \int_0^\tau a_{ij}(\tau - s) g_j(s)\, ds\right\} x_i \theta(\tau) \exp(-\mu\tau) \frac{d\sigma\, d\tau}{R}. \end{aligned} \tag{4.4}$$

We supply some further weighted arithmetic-geometric mean inequalities:

$$\begin{aligned} &\int_0^t \int_\Omega \{Q\theta + \alpha_k g_k\}(\tau) \exp(-\mu\tau)\, dx\, d\tau \\ &\qquad \le w_1 H(t. R) + \tfrac{1}{4}\kappa \int_0^t g(\tau, R) \exp(-\mu\tau)\, d\tau + \frac{Q\,\epsilon R^3}{\mu}\left(\frac{1}{2w_1 c_m} + \frac{1}{\kappa}\right); \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\int_0^t \int_\sigma \kappa_{ij} g_j(\tau)\theta(\tau) x_i \exp(-\mu\tau) \frac{d\sigma\, d\tau}{R} \\ &\qquad \le \frac{1}{2}\int_0^t \int_\sigma \left\{w_2 c\theta^2(\tau) + \frac{3A}{w_2 c_m} g_i(\tau) g_i(\tau)\right\} \exp(-\mu\tau)\, d\sigma\, d\tau; \end{aligned} \tag{4.6}$$

$$\int_0^t \int_\sigma \left\{-\alpha_i(\tau) + \int_0^\sigma \alpha_{ij}(\tau - s)g_j(s)\,ds\right\} x_i\theta(\tau) \exp(-\mu\tau) \frac{d\sigma\,d\tau}{R}$$
$$\leq \frac{1}{2} \int_0^t \int_\sigma \left\{\left(\frac{T}{w_3} + w_4\right)c\theta^2(\tau) + \frac{3A}{\mu c_m} w_3 g_i(\tau)g_i(\tau)\right\} \exp(-\mu\tau)\,d\sigma\,d\tau + \frac{Q\,\epsilon\,R^2}{2\mu c_m w_4}, \tag{4.7}$$

where w_1, w_2, w_3 and w_4 are four positive weights.

We can now prove the following result:

Theorem 3. If $\theta(t)$ is a classical solution of $\mathfrak{M}$ in $\mathcal{E}$ subject to conditions (2.8), (2.9) and (4.1)–(4.3), then

$$\int_0^t \int_\mathcal{E} \theta^2(\tau)\,dx\,d\tau \leq K\epsilon^\nu, \qquad t \in [0, T), \tag{4.8}$$

where $K = K(T)$ is a positive constant and $0 < \nu < \frac{1}{2}$.

Proof. If we take

$$w_2 = 2\left(\frac{A}{\kappa c_m}\right)^{1/2}; \qquad w_3 = \frac{2T}{w_2}; \qquad w_4 = \tfrac{1}{2}w_2, \tag{4.9}$$

and

$$\mu = \max\left\{w_1 + \frac{2A}{c_m}(1 + T), \frac{36A^2T}{\kappa^2}, 6T, \frac{1}{R_m}\right\}, \tag{4.10}$$

then inequalities (4.4)–(4.7) yield after a further integration over $[0, t]$:

$$\frac{dE}{dR} - p_1E + Qp_1T\left\{1 + \frac{1}{\mu}\left(\frac{1}{2w_1c_m} + \frac{1}{\kappa}\right) + \frac{1}{w_2c_m}\right\}\epsilon R^3. \tag{4.11}$$

Hence, on integrating this differential inequality over $[R, 2R]$ and applying the hypothesis (4.1), we can write

$$E(t, R) \leq C\exp((p - p_1)R) + 8Qp_1T\left\{1 + \frac{1}{\mu}\left(\frac{1}{2w_1c_m} + \frac{1}{\kappa}\right) + \frac{1}{w_2c_m}\right\}\epsilon\,R^3. \tag{4.12}$$

Thus, since $p_1 > p$ by assumption and $H(t, R) \leq E(t, R)$ is a nondecreasing function of R for fixed t, we can compute a constant k such that

$$H(t, R) \leq k\epsilon\frac{R^3}{a^3}, \qquad t \in [0, T), \tag{4.13}$$

where a has the dimensions of length. Finally, using inequality $(2.8)_1$ and definitions (2.11) and (2.12), we have

$$\int_0^t \int_\Omega \theta^2(\tau)\,dx\,d\tau \leq K\epsilon\exp(\mu T)\frac{R^3}{a^3}, \qquad t \in [0, T), \tag{4.14}$$

where $K = 2k/c_m$. We put

$$\mu = -\frac{(1 - 2\nu)}{T}\log\epsilon; \qquad R_M = a\epsilon^{-\nu/3}; \tag{4.15}$$

for $0 < \nu < \frac{1}{2}$ and then (4.14) and (4.15) together imply inequality (4.8) as $R \to \infty$. For ϵ small enough definition $(4.15)_1$ ensures that μ is large enough to satisfy Eq. (4.10) and definition $(4.15)_2$ ensures that $R_M > R_m$. This completes the proof of Theorem 3.

Theorem 3 clearly leads to:

THEOREM 4. If $\theta(t)$ is a classical solution of $\mathfrak{M}$ subject to conditions (2.8), (2.9), (4.1), (4.2) and zero "data", then the solution is identically zero in $\bar{\mathcal{E}} \times [0, T)$.

Proof. We notice that for zero "data" then $\epsilon = 0$ in (4.3). Furthermore, inequality (4.8) simplifies to give

$$\int_0^t \int_{\mathcal{E}} \theta^2(\tau)\, dx\, d\tau \leq 0, \qquad t \in [0, T). \tag{4.16}$$

Thus, we conclude that $\theta(t) = 0$ throughout $\bar{\mathcal{E}} \times [0, T)$.

We note that Theorem 4 is a new uniqueness result for the linearized version of the theory in [1, 3]. Although this theorem is valid for the exterior of a bounded domain in $\mathbf{R}^3$ it fails to generalize Theorem 1 of [8] which proves uniqueness for the nonlinear counterpart of $\mathfrak{M}$ on a *bounded* domain. However, the Protter method is not restricted to linear problems (see for example Beevers [14]) and it may be that further extensions of Theorem 4 for nonlinear heat flow problems are possible.

It is also worthy of note that results similar to Theorems 3 and 4 can be established for the linear theory proposed by Gurtin and Pipkin [2]. The method as described above is the same but restrictions (2.8) are replaced by

$$c \geq c_m > 0; \qquad \alpha_{ij}(0) = \alpha_{ji}(0); \qquad \alpha_{ij}(0)\eta_i\eta_j \geq \alpha\eta_i\eta_i > 0; \qquad \boldsymbol{\kappa} = \mathbf{0}. \tag{4.17}$$

It is also necessary to extend the boundedness assumptions in (2.9) to include $\ddot{\beta}$, $\dot{\alpha}$, $\ddot{\alpha}$ and the matrix with the components $\dot{\alpha}_{1i}\dot{\alpha}_{1j}$. Conditions (4.17) are the same as those required to prove a series of uniqueness theorems for the Gurtin and Pipkin theory by Herrmann and Nachlinger [7, 8]. However, these results in [7, 8] are valid for nonlinear tenperature deviations as well as any unbounded domain. The only advantage, then, of the present method over that adopted in [7, 8] would seem to be in the form of the boundary conditions. For the Protter-Graffi method allows genuine mixed boundary conditions without any restriction on the energy rate. However, Herrmann and Nachlinger [8], using the wave propagation property, establish a uniqueness theorem for solutions of a nonlinear version of $\mathfrak{M}$ provided that the energy rate is independent of the present value of the temperature. This restriction on $\mathfrak{M}$ is removed in [7] but only if the temperature is prescribed on the whole boundary.

Acknowledgement. The author would like to thank Professor R. Roy Nachlinger and L. T. Wheeler of the University of Houston for helpful criticisms of an earlier version of this paper.

REFERENCES

[1] B. D. Coleman and M. E. Gurtin, *Equipresence and constitutive equations for rigid heat conductors*, Z. Angew. Math. Phys. **18**, 199–207 (1967)

[2] M. E. Gurtin and A. C. Pipkin, *A general theory of heat conduction with finite thermal wave speeds*, Arch. Rat. Mech. Anal. **31**, 113–126 (1968)

[3] J. W. Nunziato, *On heat conduction in materials with memory*, Quart. Appl. Math. **29**, 187–204 (1971)

[4] R. R. Nachlinger and L. T. Wheeler, *Uniqueness theorems for rigid heat conductors with memory*, Quart. Appl. Math. **31**, 267–273 (1973)

[5] J. M. Finn and L. T. Wheeler, *Wave propagation aspects of the generalized theory of heat conduction*, Z. Angew. Math. Phys. **23**, 927–940 (1972)

[6] L. T. Wheeler and E. Sternberg, *Some theorems in classical elastodynamics*, Arch. Rat. Mech. Anal. **31**, 51–90 (1968)

[7] R. P. Hermann and R. R. Nachlinger, *A uniqueness theorem for the general theory of heat conduction with finite wave speeds*, Int. J. Engrg. Sci. **12**, 865–874 (1974)

[8] R. P. Herrmann and R. R. Nachlinger, *On uniqueness and wave propagation in heat conductors with memory*, J. Math. Anal. Applics. **50**, 530–547 (1975)

[9] C. E. Beevers, *Uniqueness and stability in linear visco-elasticity*, Z. Angew. Math. Phys. **26**,177–186 (1975)

[10] D. Graffi, *Sul teorema di unicità par la equazioni fluidi*, Ann. Mat. Pura Appl. **50**, 379–387 (1960)

[11] D. E. Edmunds, *On the uniqueness of viscous flows*, Arch. Rat. Mech. Anal. **14**, 171–176 (1963)

[12] J. R. Cannon and G. H. Knightly, *Continuous dependence theorems for viscous fluid motions*, SIAM J. Appl. Math. **18**, 627–640 (1970)

[13] B. Straughan, *Uniqueness and continuous dependence theorems for the conduction-diffusion solution of the Boussinesq equations on an exterior domain*, J. Math. Anal. Applics. (to appear)

[14] C. E. Beevers, *A simplified two-fluid theory for helium II*, Int. J. Engrg. Sci. **14**, 457–465 (1976)