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THEORY OF MULTI-DIFFUSION FLAMES*

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Abstract. A theory which provides categorical classification and prediction of flame configurations for multi-component mixtures consisting of N species has been developed. The application of the theory of irreducible operators decouples the species equations; and the fundamental solutions, if they exist, satisfy the same equations as that of a binary mixture. The theory is applied to two flow configurations: first, the flame at the mouth of a tube in a duct, and second, the flame in an unconfined jet. The single mode flame structure is essentially the same as that of binary diffusion approximation with proper modification of the effective diffusivity. The double mode structure consists of two diffusion modes. The individual mode satisfies the diffusion equation with characteristic effective diffusivity. The theory could be easily used by experiments and designers for practical purposes such as correlating experimental data or estimating burner size.

1. Introduction. The aerothermochemical aspects of diffusion flames have been extensively studied during the past few decades and are summarized in [1, 2], However, in the majority of the above analytical investigations, the assumption has been made that the diffusion of chemical species is a binary diffusion process. The celebrated work of Burke and Schumann [3], among others, is a classical example. Burke and Schumann presented a simple method of predicting the shape and height of the diffusion flame using a linearized theory. Indeed, the results of their analysis agree well with some experimental observations, yet there are many diffusion flames whose shape and height cannot be predicted from their theories. Among other reasons, we conjecture that this may be attributed to one of the basic assumptions made in their analysis, namely, that the binary diffusion coefficients of each interdiffusion gas pair are the same.

In practice, the diffusion flames are controlled by complicated coupled diffusion [4, 5, 6] of all the species present in the flame. Under a multi-diffusion process, the rate of diffusion of each species is interrelated with that of the others. Hence, the overall diffusional pattern, the rates of multi-step chemical reactions, locations of flames together with velocity and temperature profiles predicted under the model of multi-diffusion are different from what would be predicted by the binary diffusion approximation. Unfortunately, the detailed physical processes and the physical consequences of multi-diffusion processes in many reacting systems are not well understood to date.

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The purpose of the present paper is to develop a theory which provides categorical classification and investigations of multi-diffusional flames consisting of N species, and prediction of the flame configurations for these various classes of coupled diffusion flames.

It may be added that the major difficulties associated with multi-component systems are the prediction of the overall fields which are governed by flow processes, chemical reactions, and transparent processes. Various analytical procedures which permit simpler mathematical treatment have been developed for a large class of reacting flow systems within the binary diffusion approximation [1, 2]. An analytical procedure which would simplify the prediction of the complete flow field for a multi-diffusing reacting system is anticipated to be somewhat complicated and will not be presented in the present paper. Nevertheless, the present analytical scheme incorporated with existing techniques provides a simpler mathematical scheme and interesting physical insight into the complete flow field of the reacting gas system. A detailed theoretical investigation which addresses to the multi-diffusion flame stabilized in a reacting boundary layer is presently being made and will be reported elsewhere in the future.

Within the flame sheet approximation [1, 2, 3, 7] the analysis, which is greatly facilitated by the application of the theory of irreducible operators [8, 9, 10], reveals that the multi-diffusion flames may be classified in two groups.

The first group includes those flames whose configurations are determined by M fundamental modes ($1 \leq M \leq N$) which are described by diffusion equations of the binary type. The flame configurations of the second group are determined, in general, by coupled multi-diffusional modes described by homogeneous higher-order partial differential equations. The order of the equations does not exceed $2(N - 1)$.

The physical parameters that determine the cetegory of a given multi-diffusional flame are the diffusivity matrix and the stoichiometric coefficients of the gaseous mixture composing the flame.

For example, in a flame composed of fuel, oxidizer, a combustion product, and an inert gas, the flame configuration may be determined by one or two basic diffusion modes. The actual number of basic diffusion modes depends on the conditions imposed on the relations between the diffusivity matrix and stoichiometric coefficients. If the conditions for the existence of fundamental modes cease to exist, the flame configuration must be determined by higher-order diffusion equations.

The theory developed is applied to the multi-diffusion flame at the mouth of a tube in a duct and to the flame of a laminar jet. For ternary and quarternary systems, formulas for determining flame shape and height are obtained which could easily be used by experimentalists and designers for correlating experimental data or estimating burner size. The applications to other flow geometries are possible under the same assumptions.

2. Analysis. For a multi-component system of N species the steady-state species equations are

$$\rho \mathbf{v} \cdot \nabla Y_i = \dot{w}_i - \nabla \cdot \mathbf{J}_i , \qquad i = 1, 2, \cdots, N \tag{1}$$

where $\dot{w}_i$ is the production term. If the chemical changes occur by a single reaction step

$$\nu_F F + \nu_0 O + \sum_{i=3}^{N} \nu_i' M_i \rightarrow \sum_{i=3}^{N} \nu_i'' M_i , \tag{2}$$

where F is the symbol for fuel, O is the symbol for oxydizer and M is the symbol for

chemical species i, then

$$\dot{w}_i = W_i(\nu_i'' - \nu_i')\omega, \qquad i = 1, 2, \cdots, N. \tag{3}$$

The mass flux vector, $\mathbf{J}_i$, is given by Eq. (8.1-1) of [11, p. 516]. Assuming the thermal diffusion, pressure diffusion and body force are negligible, we have

$$\mathbf{J}_i = -\rho \sum_{j=1}^{N-1} \hat{D}_{ij} \nabla Y_j \qquad i = 1, 2, \cdots, N \tag{4}$$

where

$$\hat{D}_{ij} = \frac{W_i}{W}(D_{iN} - D_{ij}) - W_i\left(\sum_{K=1}^{N} D_{iK}Y_K\right)\left(\frac{1}{W_N} - \frac{1}{W_j}\right) \tag{5}$$

We note that $\hat{D}_{ij}$, which we shall call the modified multi-component diffusion coefficient, depends on the mass fraction Y_i collectively, i.e., through the molecular weight of the mixture, W, and $(\sum_{k=1}^{N} D_{ik}Y_k)$. One of the important differences between the mass flux vector of multi-component systems, i.e. Eq. (4), and that of binary systems, i.e. Fick's law, is that Eq. (4) depends not only on the gradient of its own concentration but also on the concentration gradient of other species and the corresponding diffusivities.

Substituting Eq. (4) into Eq. (1) and making use of the fact that

$$\sum_{j=1}^{N} Y_j = 1, \tag{6}$$

we have

$$\rho\mathbf{v}\cdot\nabla Y_i = \sum_{j=1}^{N-1} \nabla\cdot(\rho\hat{D}_{ij}\nabla Y_j) + W_i(\nu_i'' - \nu_i')\omega, \qquad i = 1, 2, \cdots, (N-1). \tag{7}$$

For convenience, we introduce the following notations

$$\begin{aligned} Z_i &= \frac{\pm Y_i}{W_i(\nu_i'' - \nu_i')} \\ d_{ij} &= \pm\hat{D}_{ij}\cdot\frac{W_j(\nu_j'' - \nu_j')}{W_i(\nu_i'' - \nu_i')} \end{aligned} \qquad \text{if} \quad \nu_i'' - \nu_i' \neq 0, \tag{8}$$

where $+$ sign for reactant, $-$ sign for products, and

$$\begin{aligned} Z_i &= Y_i/W_i \\ d_{ij} &= \hat{D}_{ij}\frac{W_j(\nu_j'' - \nu_j')}{W_i} \end{aligned} \qquad \text{if} \quad \nu_i'' - \nu_i' = 0. \tag{9}$$

Thus Eq. (7) becomes

$$\rho\mathbf{v}\cdot\nabla Z_i = \sum_{j=1}^{N-1} \nabla\cdot(\rho d_{ij}\nabla Z_j) + \sigma_i\omega \tag{10}$$

where σ_i is -1 for reactant, $+1$ for products and 0 for inert gas. Eqs. (10) are a coupled system of $(N-1)$ equations. In general, one must either solve a $2(N-1)$th-order equation after elimination of other dependent variables, or to perform a matrix inversion in order to obtain second derivatives of dependent variables, $\nabla^2 Z_i$, explicitly. Neither of these procedures is a simple exercise, though they can be done in principle. However,

a set of fundamental solutions governed by a system of uncoupled diffusion equations can be obtained by the application of the so-called irreducible operator Chiu [8] has developed. The method has been successfully applied to many coupled transport problems [8, 9, 10]. We first apply the linear transformation of the dependent variables; i.e., let

$$Z = \sum_{j=1}^{N-1} \alpha_{ij}\psi_j \tag{11}$$

where $\bar{\alpha}$ is in general a $(n-1) \times (n-1)$ matrix containing $(n-1) \times (n-1)$ arbitrary elements. Substituting Eq. (11) into Eq. (10), we have

$$\rho\mathbf{v}\cdot\nabla(\bar{\alpha}\psi) = \nabla\cdot(\rho\bar{d}\nabla(\bar{\alpha}\bar{\psi})) + \mathfrak{d}\omega; \tag{12}$$

multiplying Eq. (12) by the inverse of $\bar{\alpha}$ from the left, we have

$$\rho\mathbf{v}\cdot\nabla\psi = \nabla\cdot(\bar{L}\nabla\bar{\psi}) + \mathbf{g}\omega \tag{13}$$

where

$$\bar{L} = \rho(\bar{\alpha}^{-1}\,\bar{d}\bar{\alpha}) \tag{14}$$

and

$$\mathbf{g} = \bar{\alpha}^{-1}\bar{\sigma}. \tag{15}$$

By a proper choice of α, Eq. (13) can be reduced to the following form in which $\bar{L}$ is diagonalized:

$$\rho\mathbf{v}\cdot\nabla\psi_i = L_{ii}\nabla^2\psi_i + g_i\omega. \tag{16}$$

The solutions ψ_i $(i = 1, 2, \cdots, N-1)$ of Eq. (16) are called the fundamental mode of the system.

Single mode flame structure. If the diffusivity matrix $\bar{d}$ is such that there exists a matrix $\bar{\alpha}$ which satisfies the following conditions

(a) $L_{1j} = 0, \quad j \neq 1$ (17)

(b) $g_1 = 0$

(c) $\nu_{11} = \nu_{12} \neq 0$ and $\nu_{1j} = 0, \quad j \geq 3$

where ν_{ij} are the elements of inverse of $\bar{\alpha}$, then the fundamental mode, ψ_1, is described by a homogeneous second-order partial differential equation

$$\rho\mathbf{v}\cdot\nabla\psi_1 = L_{11}\nabla^2\psi_1 \tag{18}$$

and

$$\psi_1 = \nu_{11}(Z_1 - Z_2) = \nu_{11}\left[\frac{Y_F}{\nu_F' W_F} - \frac{Y_0}{\nu_0' W_0}\right]. \tag{19}$$

Flame shape is determined by one fundamental mode ψ_1 :

$$\psi_1(\mathbf{r}_f) = 0 \tag{20}$$

or

$$Y_F(\mathbf{r}_f)/Y_0(\mathbf{r}_f)\cdot W_0/W_F = \nu_F'/\nu_0'. \tag{21}$$

Eq. (21) states the well-known fact that at flame sheet the ratio of number of moles of the fuel to that of the oxidizer is exactly the stoichiometric ratio. This is the flame

composed of a single mode. If we extend this idea, we would get the class of diffusion flames that are composed of double and triple modes.

Double mode flame structures. If the diffusivity matrix is such that there exists a matrix $\bar{\bar{\alpha}}$ which satisfies the following conditions

$$\begin{aligned} &\text{(a)} && L_{ij} = L_{2,j+1} = 0, && j \neq 1 \\ &\text{(b)} && g_1 = g_2 = 0 \\ &\text{(c)} && \nu_{ij} = \nu_{2j} = 0, && j \geq 3, \end{aligned} \tag{22}$$

then the fundamental modes, ψ_1 and ψ_2, are described by second-order partial differential equations:

$$\rho \mathbf{v} \cdot \nabla \psi_1 = L_{11} \nabla^2 \psi_1 \tag{23}$$

$$\rho \mathbf{v} \cdot \nabla \psi_2 = L_{22} \nabla^2 \psi_2 \tag{24}$$

and flame shape is determined by two modes ψ_1 and ψ_2.

M fundamental mode flame structure, $M \leq N - 1$. If the diffusivity matrix $\bar{\bar{d}}$ is such that there exists a matrix $\bar{\bar{\alpha}}$ which satisfies the following conditions

$$\begin{aligned} &\text{(a)} && L_{ij} = 0 && i \leq M, && j \geq 2 \\ &\text{(b)} && g_i = 0 && i \leq M \\ &\text{(c)} && \nu_{ij} = 0 && i \leq M, && j > M, \end{aligned} \tag{26}$$

then

$$\rho \mathbf{v} \cdot \nabla \psi_i = L_{ii} \nabla^2 \psi_i \qquad i \leq M \tag{27}$$

and flame shape is determined by M fundamental modes ψ_1, $i = 1, 2, \cdots, M$:

$$\begin{vmatrix} \psi_1 & \nu_{12} & \cdots & \nu_{1M} \\ \psi_2 & \nu_{22} & \cdots & \nu_{2M} \\ \vdots & \vdots & \vdots & \vdots \\ \psi_M & \nu_{M2} & \cdots & \nu_{MM} \end{vmatrix} - \begin{vmatrix} \nu_{11} & \psi_1 & \nu_{13} & \cdots & \nu_{1M} \\ \nu_{12} & \psi_2 & \nu_{23} & \cdots & \nu_{2M} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \nu_{1M} & \psi_M & \nu_{M3} & \cdots & \nu_{M} \end{vmatrix} = 0. \tag{28}$$

Thus, we have completed the formal theory of fundamental modes of flame structures. Next, we shall illustrate the previous classification with the examples of binary, ternary, and quaternary systems.

Binary system. The formal theory we have developed previously will be applied to the simplest system of binary diffusion of two non-reacting gases, e.g. the binary diffusion of two initially separated noble gases. In this case, the diffusion coefficients take the following values

$$\hat{D}_{11} = \hat{D}_{22} = 0 \quad \text{and} \quad \hat{D}_{12} = \hat{D}_{21} = D \tag{29}$$

where D is the binary diffusion coefficient. Then, as one would expect, Eq. (17) reduces to the equation of binary diffusion:

$$\rho \mathbf{v} \cdot \nabla Y_1 = -\nabla \cdot (\rho D \nabla Y_1) \tag{30}$$

Ternary system. The simplest nontrivial application of the theory is the diffusion flame of a ternary system which composed of fuel, oxidizer and compustion product. In that case, Eq. (2) takes the following form:

$$\nu_F F + \nu_0 0 \rightarrow \nu_p P. \tag{31}$$

The combustion of hydrogen and oxygen at moderate pressure and temperature falls on this category. Subscripts 1, 2, and 3 refer to, fuel, oxygen and product, respectively. Thus Eq. (10) becomes

$$\rho \mathbf{v} \cdot \nabla Z_1 = \nabla \cdot (\rho d_{11} \nabla Z_1 + \rho d_{12} \nabla Z_2) + \omega, \tag{32}$$

$$\rho \mathbf{v} \cdot \nabla Z_2 = \nabla \cdot (\rho d_{21} \nabla Z_1 + \rho d_{22} \nabla Z_2) + \omega. \tag{33}$$

If the elements of the diffusivity matrix satisfy the following condition:

$$d_{11} + d_{12} = d_{21} + d_{22} \tag{34}$$

then we can choose the matrix $\bar{\bar{\alpha}}$ as

$$\bar{\bar{\alpha}} = \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 1 \end{pmatrix} \tag{35}$$

and the fundamental mode, ψ_1, is governed by the following equation:

$$\rho \mathbf{v} \cdot \nabla \psi_1 = \rho (d_{11} - d_{21}) \nabla^2 \psi_1 . \tag{36}$$

The configuration of the diffusion flame of a single mode can be calculated by solving the algebraic solution of Eq. (20).

Quarternary system. From the practical point of view, the reactants of the combustion are usually diluted initially with some noble gases or nitrogen, and the diffusion of quaternary system arises. Letting subscripts 1, 2, 3, and 4 denote fuel, oxidizer, product and inert gases, respectively, we have the equations of the form of Eq. (10) with $N = 4$.

If the elements of the diffusivity matrix satisfy the following condition:

$$d_{13} = d_{23}, \qquad d_{11} + d_{12} = d_{21} + d_{22} \tag{37}$$

then we can choose the matrix $\bar{\bar{\alpha}}$ as

$$\bar{\bar{\alpha}} = \begin{pmatrix} \frac{1}{2} & 1 & 1 \\ -\frac{1}{2} & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{38}$$

and the fundamental mode, ψ_1, exists:

$$\rho \mathbf{v} \cdot \nabla \psi_1 = \rho (d_{11} - d_{12}) \nabla^2 \psi_1 . \tag{39}$$

If the elements of the diffusivity matrix satisfy the following conditions

$$d_{11} + d_{12} - d_{13} = d_{21} + d_{22} - d_{23} = d_{33} - d_{31} - d_{32} \tag{40}$$

$$d_{13} \neq d_{23} \tag{41}$$

then two fundamental modes, ψ_1 and ψ_2, exist. The corresponding matrix $\bar{\bar{\alpha}}$ is chosen

as follows:

$$\bar{\bar{\alpha}} = \begin{bmatrix} 1 & 1 & 1 \\ E_+ & E_- & 1 \\ 0 & 0 & -1 \end{bmatrix} \tag{42}$$

where E_+ and E_- are the roots of the quadratic equation.

$$(d_{12} + d_{32})E^2 + (d_{11} - d_{22} + d_{31} - d_{32})E - (d_{21} + d_{31}) = 0 \tag{43}$$

The equations describing ψ_1 and ψ_2 are Eq. (23) and Eq. (24), respectively. The corresponding L_{11} and L_{22} are

$$L_{11} = (d_{22} + d_{32})E_+ - (d_{11} + d_{31})E_- + 2(d_{21} + d_{31}) \tag{44}$$

$$L_{22} = (d_{11} + d_{31})E_+ - (d_{22} + d_{32})E_- - 2(d_{21} + d_{31}) \tag{45}$$

The flame shape is determined by

$$(1 - E_+)\psi_1 + (1 - E_-)\psi_2 = 0 \tag{46}$$

The features we have been discussing previously are typical of a system whose diffusivities, stoichimetric coefficients and molecular weights satisfy certain conditions. What about the systems which possess no fundamental modes? Such flames are apparently more complex. In fact, the flame shape of such a system can be shown to be governed by uncoupled higher-order differential equations. The order of such equations depends on the diffusivity, the stoichimetric coefficients, and the molecular weights and cannot exceed $2(N - 1)$.

3. Applications. We shall now apply the theory developed previously to different flow geometries. The first problem is the flame at the mouth of a tube in a duct. Both the axisymmetric and two-dimensional cases will be examined. The second problem is the flame of an unconfined jet.

The flame at the mouth of a tube in a duct. The basic assumptions and the equations describing the present problem are essentially the same as those adopted by Burke and Schumman [3] except for the approximation of binary diffusion. The Burke and Schumann problem is illustrated in Figure 1; fuel issues from a tube into a concentric duct through which oxidizer is flowing. A flame is established at the mouth of the tub. The flame height is required for the design of burners of this type.

Following [3], we assume the velocity is parallel to the axis of the tube everywhere, the total mass flow in the z-direction is constant everywhere in the duct, the effective diffusivity L_{ii} is constant throughout the duct, and the axial diffusion is negligible in comparison with radial diffusion.

Under these assumptions, the single-mode equation for ternary systems, Eq. (36) and quaternary systems, Eq. (39) become

$$\frac{\rho v}{L_{11}} \frac{\partial \psi_1}{\partial z} - \frac{1}{r^\epsilon} \frac{\partial}{\partial r}\left(r^\epsilon \frac{\partial \psi_1}{\partial r}\right) = 0 \tag{47}$$

with initial conditions

$$\psi_1 = Y_{F,0}/W_F\nu_F \quad \text{at} \quad z = 0, \qquad 0 \leq r < a \tag{48}$$

$$\psi_1 = Y_{0,0}/W_0\nu_0 \quad \text{at} \quad z = 0, \qquad a < r < b \tag{49}$$

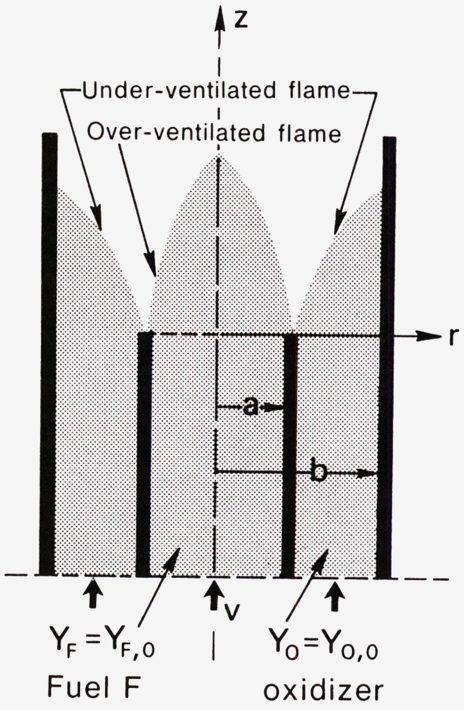

FIG. 1. Diagram of the Burke-Schumann problem.

and boundary conditions

$$\partial\psi_1/\partial r = 0 \qquad \text{at} \quad r = b \tag{50}$$

$$\psi_1 = \text{finite} \quad \text{at} \quad r = 0 \tag{51}$$

where $\epsilon = 0$ for two-dimensional and $\epsilon = 1$ for axisymmetric flow. By the method of separation of variables, it can be shown that for a cylindrical flame

$$\psi_1(r, z) = +\left(\frac{Y_{F,0}}{W_F\nu_F} - \frac{Y_{0,0}}{W_0\nu_0}\right)\frac{a}{b} + \frac{Y_{F,0}}{W_F\nu_F} - 2\left(\frac{Y_{F,0}}{W_F\nu_F} + \frac{Y_{0,0}}{W_0\nu_0}\right)$$

$$\times \frac{a}{b}\sum_{n=1}^{\infty}\frac{1}{\mu_n}\frac{\left(J_1\dfrac{a}{b}\mu_n\right)}{J_0^2(\mu_n)}J_0\left(\frac{r}{b}\mu_n\right)\exp\left[-\frac{L_{11}z}{\rho v b^2}\mu_n^2\right] \tag{52}$$

where J_0 and J_1 are Bessel functions of the first find (of order 0 and 1, respectively), and μ_n represents successive roots of the equation $J_1(\mu) = 0$ (with the ordering convention $\mu_n > \mu_{n-1}, \mu_0 = 0$). For a flat flame

$$\psi_1(x, z) = -\frac{Y_{F,0}}{W_F\nu_F} + \frac{Y_{0,0}}{W_0\nu_0}\frac{a}{b} + \frac{Y_{F,0}}{W_F\nu_F} - \frac{2}{\pi}\frac{Y_{F,0}}{W_F\nu_F}$$

$$+ \frac{Y_{0,0}}{W_0\nu_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\pi\frac{a}{b}\right)\cos\left(n\pi\frac{x}{b}\right)\exp\left[-\frac{L_{11}z}{\rho v b^2}n^2\pi^2\right] \tag{53}$$

The flame surface is determined by Eq. (20).

The double mode structure for quaternary systems satisfies some equations as Eq. (47) with initial conditions at $z = 0$, $0 \leq r < a$:

$$\psi_1 = \frac{-E_-}{E_+ - E_-} \frac{Y_{F,0}}{W_F \nu_F} \tag{54}$$

$$\psi_2 = \frac{E_+}{E_+ - E_-} \frac{Y_{F,0}}{W_F \nu_F}\ ; \tag{55}$$

at $z = 0$, $a < r < b$

$$\psi_1 = \frac{-1}{E_+ - E_-} \cdot \frac{Y_{0,0}}{W_0 \nu_0} \tag{56}$$

$$\psi_2 = \frac{1}{E_+ - E_-} \cdot \frac{Y_{0,0}}{W_0 \nu_0} \cdot \tag{57}$$

The solutions for ψ_1 and ψ_2 are of the same form as Eq. (52) and (53) with proper modifications of the initial conditions according to Eq. (54–57).

The flame of an unconfined jet. The first attempt at the theoretical treatment of the problem is perhaps due to Fay (11). We shall follow Fay in the following analysis, except for the assumptions of binary diffusion. The equation of motion of a laminar jet assuming no pressure gradients or body force is, for two-dimensional and axially symmetric motion respectively,

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \tag{58}$$

$$\rho v \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r}\left(\mu r \frac{\partial u}{\partial r}\right) \cdot \tag{59}$$

The equations of fundamental modes are, for two-dimensional and axially symmetric motion respectively,

$$\rho u \frac{\partial \psi_i}{\partial x} + \rho v \frac{\partial \psi_i}{\partial y} = \frac{\partial}{\partial y}\left(L_{ii} \frac{\partial \psi_i}{\partial y}\right), \tag{60}$$

$$\rho v \frac{\partial \psi_i}{\partial x} + \rho v \frac{\partial \psi_i}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r}\left(L_{ii} r \frac{\partial \psi_i}{\partial r}\right) \cdot \tag{61}$$

Under the condition that $L_{11} = \mu$ it can readily be seen that a solution of the fundamental modes, Eq. (3), are

$$\psi_1 = B + Cu \tag{62}$$

where B and C are arbitrary constants determined by the boundary conditions.

For single-mode structure, we have

$$\psi_1 = \frac{-U_e}{U_c - U_e}\left[\frac{Y_{F,e}}{W_F \nu_F} + \frac{Y_{0,e}}{W_0 \nu_0} \cdot \frac{U_e}{U_e}\right] + \frac{1}{U_c - U_e}\left[\frac{Y_F}{W_F \nu_F} + \frac{Y_0}{W_0 \nu_0}\right] u \tag{63}$$

where subscripts c, e refer to the value at the axis and ambient, respectively. The flame surface is determined by Eq. (20). For fundamental modes other than single mode, the condition that all L_{ii} are equal to μ reduces to the case of binary approximation. To obtain the multi-diffusion effect, we have to abandon the condition. The resulting equation

could only be solved numerically. However, if one is interested in the flame height only, analytical solution is possible with some reasonable approximation. Near the axis of the circular jet the velocity, according to the similar solution [12, 13], is

$$U_c = (3/8\pi)(K/\mu x) \tag{64}$$

where K is the momentum of the jet.

For an over-ventilated flame, the combustion is controlled by the diffusion of oxygen toward the axis. Near the axis v is negligible and the second term on the left-hand side of Eq. (61) can be dropped. The resulting equation becomes

$$\frac{3}{8\pi}\frac{K}{\nu x L_{ii}}\frac{\partial \psi_i}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_i}{\partial r}\right) \tag{65}$$

With initial conditions Eq. (54–57) and boundary conditions Eq. (50–51), the double-mode solutions to Eq. (65) are

$$\psi_1 = \psi_{i,e} = (\psi_{i,c} - \psi_{i,e})\int_0^\infty \exp(-\omega^2\xi_i)J_0(\omega\bar{r})J_1(\omega)\,d\omega \tag{66}$$

where $\psi_{1,c}$ and $\psi_{1,e}$ are Eq. (54–55) and Eq. (56–77), respectively. The radius is normalized by the jet radius, and

$$\xi_i = \frac{4}{3}\frac{\pi\nu L_{ii}}{Ka^2}x^2. \tag{67}$$

At the axis, i.e., $\bar{r} = 0$, Eq. (66) becomes

$$\psi_i - \psi_{i,e} = (\psi_{i,c} - \psi_{i,e})\left[1 - \exp\left(-\frac{1}{4\xi_i}\right)\right] \tag{68}$$

Substituting Eq. (68) into Eq. (46), we have the flame height.

4. Concluding remarks. A theory which predicts the shape of multi-component diffusion flames has been developed. Applications have been made to Burke and Schumann problems and laminar jet diffusion flame. For binary and quarternary systems formula for predicting flame shape have worked out for both cylindrical and flat flames. These formulas could be easily used by experimentalists and designers for correlating experimental data or estimating burner size. The theory could be applied to other flow configurations, e.g., the flame of droplet burning, the flame of a flat plate, the flame at a stagnation point and flame of a counter jet. The theory could also be used for selecting the proper diffusivities for the binary approximation, e.g., in comparing the binary diffusivity of each pair with the proper effective diffusivity L_{ii}, the one which is closest to L_{ii} should be the pair of diffusivity.

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