BENDING WAVES IN SHELLS*

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1. Introduction. Most of the approximations and more formal asymptotic expansions which have been used for the analysis of the static and dynamic response of thin, elastic shells are analogous to the asymptotic methods that have been developed in wave mechanics. Whitham [10] has developed a variational approach for the determination of the significant features of a wave propagating through an inhomogeneous, dispersive medium. In the present paper, it is shown that this variational approach can be utilized for waves in shells and provides a dramatic simplification in the determination of the amplitude function.

Asymptotic analysis of shells began with the work of H. Reissner, O. Blummenthal, and E. Meissner (1912–1915). Direct application of asymptotic wave analysis was utilized in the discussion of transition points occurring in the axisymmetric motion of shells of revolution by Ross [7]. For the general surface, the asymptotic analysis leads to "geometric optics", in which the determination of "rays" and "caustics" plays a vital role. Steele [9] discusses the fundamental "point load" solution for the shell with high prestress and negligible bending stiffness. Germogenova [1] obtains the wave solution from the shallow-shell equations, which include the bending stiffness but not prestress. Generally, in wave mechanics, the frequency is used as the large parameter, but for shells, the convenient parameter is the radius-to-thickness ratio. Thus the geometric optics analysis is useful for the static problem, as discussed by Steele [8]. The Airy function solution for a region containing a caustic is given by Logan [3] and the Bessel function fundamental solution is given by Prat [5].

In the present paper, we consider wave propagation on the general shell surface with prestress included and without any *a priori* assumption concerning "shallowness". Attention is restricted to waves which have a wavelength of the order of magnitude of the square root of thickness times radius. These can be clearly identified as "bending" waves in the special case of axisymmetric deformation of the shell of revolution, but generally include membrane and inextensional effects. For the cylindrical shell, the present solution gives exactly the well-known solution of Donnell's equations for vibration and (classical) stability. Excluded from our consideration are waves with wave speeds of the order of the shear velocity in the shell material—either the extremely long wavelength membrane waves or the extremely short wavelength transverse shear waves. For many, if not most practical problems, these are of minor significance.

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2. Shell equations. A thorough discussion of classical and modern theories for the behavior of thin shells is given by Naghdi [4]. For the first-approximation linear theory for the isotropic elastic shell the following equations are relevant.

The shell mid-surface, with position vector $\mathbf{r}$, has the metric and curvature tensors

$$\boldsymbol{\delta} = a_{\alpha\beta}\mathbf{a}^{\alpha} \otimes \mathbf{a}^{\beta}, \ \mathbf{b} = b_{\alpha\beta}\mathbf{a}^{\alpha} \otimes \mathbf{a}^{\beta} \tag{2.1}$$

where $\mathbf{a}^{\alpha}$ are the reciprocal base vectors. Also useful is the rotation tensor (for an oriented surface)

$$\boldsymbol{\varepsilon} = a(\mathbf{a}^1 \otimes \mathbf{a}^2 - \mathbf{a}^2 \otimes \mathbf{a}^1) \tag{2.2}$$

where

$$a = (\mathbf{a}_1 \times \mathbf{a}_2)\cdot\mathbf{a}_3 = (a_{11}a_{22} - a_{12}^{\ 2})^{1/2} \tag{2.3}$$

in which $\mathbf{a}_3$ is the unit normal to the surface.

The displacement vector is

$$\mathbf{v} = v^{\alpha}\mathbf{a}_{\alpha} + w\mathbf{a}_3 \tag{2.4}$$

the gradient of which is

$$\nabla\mathbf{v} = \boldsymbol{\gamma} - \boldsymbol{\psi} \otimes \mathbf{a}_3 . \tag{2.5}$$

The midsurface strain measure is the symmetric part of $\boldsymbol{\gamma}$,

$$\hat{\boldsymbol{\gamma}} = \tfrac{1}{2}(\boldsymbol{\gamma} + \boldsymbol{\gamma}^t), \tag{2.6}$$

while $\boldsymbol{\psi}$ provides the rotation of the normal. The change in curvature of the surface depends on the intrinsic part of the gradient of $\boldsymbol{\psi}$

$$\boldsymbol{\kappa} = \nabla\boldsymbol{\psi}\cdot\boldsymbol{\delta}. \tag{2.7}$$

However, the appropriate bending strain measure is the symmetric tensor

$$\hat{\boldsymbol{\kappa}} = \tfrac{1}{2}[\boldsymbol{\kappa} + \tfrac{1}{2}(\mathbf{b}\cdot\boldsymbol{\gamma} - \boldsymbol{\gamma}\cdot\mathbf{b})] + \tfrac{1}{2}[\boldsymbol{\kappa} + \tfrac{1}{2}(\mathbf{b}\cdot\boldsymbol{\gamma} - \boldsymbol{\gamma}\cdot\mathbf{b})]^t. \tag{2.8}$$

The constitutive relations between stress and strain measures are

$$\hat{\mathbf{N}} = \frac{Eh}{1-\nu^2}(\hat{\boldsymbol{\gamma}} - \nu\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{\gamma}}\cdot\boldsymbol{\varepsilon}) \tag{2.9}$$

$$\hat{\mathbf{M}} = Ehc^2(\hat{\boldsymbol{\kappa}} - \nu\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{\varepsilon}) \tag{2.10}$$

in which c is the reduced thickness

$$c^2 = h^2/12(1 - \nu^2). \tag{2.11}$$

Finally, the Lagrangian density per unit of surface area is

$$L = \tfrac{1}{2}[\rho h\dot{\mathbf{v}}\cdot\dot{\mathbf{v}} - \hat{\mathbf{N}} : \hat{\boldsymbol{\gamma}} - \hat{\mathbf{M}} : \hat{\boldsymbol{\kappa}} - \boldsymbol{\psi}\cdot\hat{\mathbf{N}}_0\cdot\boldsymbol{\psi}]. \tag{2.12}$$

In addition to the kinetic and strain-energy terms, a membrane "prestress" $\mathbf{N}_0$ has been included. The displacement (2.4) is interpreted as the small displacement from the primary equilibrium state, in which the bending stresses and rotations are negligible.

The remarkable feature of the approach developed by Whitham [10] for the asymptotic analysis of waves is that no further equations are necessary. Thus the effort required to use Novozhilov-type equations by Steele [8] and "shallow shell" equations by Germogenova [1] as the starting point of the asymptotic analysis can be completely avoided.

3. Monochromatic wave solution. The appropriate expansion for bending wave propagation is

$$\mathbf{v} = \exp(i\theta)[c\eta\mathbf{a}_3 - ic^{3/2}\boldsymbol{\mu} + O(c^2)] \tag{3.1}$$

in which the phase function is

$$\theta = \theta(\mathbf{r}, t) = -\omega t + c^{-1/2}\zeta(\mathbf{r}). \tag{3.2}$$

The spatial change in phase, given by ζ, the amplitude of normal displacement, given by η, and the amplitude of tangential displacement, given by the vector $\boldsymbol{\mu}$, are the unknown functions. It is assumed, however, that these functions are of "slow" variation, i.e. independent of the magnitude of c. Thus the exponential term in (3.1) provides the only "rapid" variation in the solution. In this section, for simplicity, E, c, and ω are all constant.

The gradient of (3.1) gives the rotation (2.5)

$$\boldsymbol{\psi} = -\nabla\mathbf{v}\cdot\mathbf{a}_3 = \exp(i\theta)[-ic^{1/2}\eta\mathbf{k} + O(c)] \tag{3.3}$$

where $\mathbf{k}$ is the gradient $\mathbf{k} = \nabla\zeta$ and the midsurface strain (2.6) is

$$\hat{\boldsymbol{\gamma}} = \exp(i\theta)[c\boldsymbol{\alpha} + O(c^{3/2})] \tag{3.4}$$

where

$$\boldsymbol{\alpha} = \tfrac{1}{2}(\mathbf{k}\otimes\boldsymbol{\mu} + \boldsymbol{\mu}\otimes\mathbf{k}) - \eta\mathbf{b}. \tag{3.5}$$

The bending strain measure (2.8) is

$$\hat{\mathbf{k}} = \exp(i\theta)[\eta\mathbf{k}\otimes\mathbf{k} + O(c^{1/2})]. \tag{3.6}$$

The stress measures (2.9), (2.10) are

$$\begin{aligned}\hat{\mathbf{N}} &= \frac{Ehc}{1-\nu^2}\exp(i\theta)[\boldsymbol{\alpha} - \nu\boldsymbol{\varepsilon}\cdot\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon} + O(c^{1/2})],\\ \hat{\mathbf{M}} &= Ehc^2\exp(i\theta)[\eta(\mathbf{k}\otimes\mathbf{k} - \nu\boldsymbol{\varepsilon}\cdot\mathbf{k}\otimes\mathbf{k}\cdot\boldsymbol{\varepsilon}) + O(c^{1/2})].\end{aligned} \tag{3.7}$$

The (real parts) of the stress and strain measures are substituted into (2.12). Following Whitham [10], the "time-averaged" Lagrangian is computed. This eliminates explicit dependence on the phase function θ:

$$\begin{aligned}\mathfrak{L} &= \frac{\omega}{\pi}\int_0^{2\pi/\omega} L\,dt\\ &= \tfrac{1}{2}Ehc^2\left[\left(\frac{\rho\omega^2}{E} - k^4 - \mathbf{k}\cdot\frac{\mathbf{N}_0}{Ehc}\cdot\mathbf{k}\right)\eta^2 - \frac{1}{1-\nu^2}(\boldsymbol{\alpha} - \nu\boldsymbol{\varepsilon}\cdot\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}) : \boldsymbol{\alpha}.\right.\end{aligned} \tag{3.8}$$

Hamilton's principle states that the integral

$$\iint L\,d\Sigma\,dt \tag{3.9}$$

must have a stationary value. After the average over time is taken, we are left with the surface integral

$$\int \mathfrak{L}\, d\Sigma \tag{3.10}$$

which must be stationary. Since $\mathfrak{L}$ depends explicitly on the unknown amplitude functions η and $\boldsymbol{\mu}$, and on the spatial derivatives of θ, the Euler equations of the variational problem are

$$\partial \mathfrak{L}/\partial \boldsymbol{\mu} = 0, \tag{3.11}$$

$$\partial \mathfrak{L}/\partial \eta = 0, \tag{3.12}$$

$$\nabla \cdot (\partial \mathfrak{L}/\partial \mathbf{k}) = 0. \tag{3.13}$$

The condition (3.11) can be used to determine $\boldsymbol{\mu}$ in terms of η, after which (3.12) will be recognized as the eikonal equation for ζ and (3.13) as the transport equation for η, in the form discussed by Hayes [2] and Whitham [10].

The direct calculation indicated by (3.11) is tedious. Instead, we note that

$$\partial \mathfrak{L}/\partial \boldsymbol{\mu} = (\partial \mathfrak{L}/\partial \boldsymbol{\alpha}) \cdot (\partial \boldsymbol{\alpha}/\partial \boldsymbol{\mu}) = 0 \tag{3.14}$$

which gives

$$(\boldsymbol{\alpha} - \nu \boldsymbol{\varepsilon} \cdot \boldsymbol{\alpha} \cdot \boldsymbol{\varepsilon}) \cdot \mathbf{k} = 0. \tag{3.15}$$

Thus from (3.7) we see that, to first approximation, $\hat{\mathbf{N}}$ must be orthogonal to the phase gradient $\mathbf{k}$. It is possible, therefore, to write

$$\hat{\mathbf{N}} = \hat{N} \boldsymbol{\varepsilon} \cdot \mathbf{k} \otimes \boldsymbol{\varepsilon} \cdot \mathbf{k} k^{-2} \tag{3.16}$$

where $\hat{N}$ is the trace of $\hat{\mathbf{N}}$. The inverse of (2.9) is

$$\hat{\boldsymbol{\gamma}} = \frac{1}{Eh} (\hat{\mathbf{N}} + \nu \boldsymbol{\varepsilon} \cdot \hat{\mathbf{N}} \cdot \boldsymbol{\varepsilon}). \tag{3.17}$$

Thus from (3.4), (3.5), and (3.16) this relation is obtained:

$$c \exp (i\theta) [\tfrac{1}{2}(\mathbf{k} \otimes \boldsymbol{\mu} + \boldsymbol{\mu} \otimes \mathbf{k}) - \eta \mathbf{b}] = \frac{\hat{N}}{Eh} [\boldsymbol{\varepsilon} \cdot \mathbf{k} \otimes \boldsymbol{\varepsilon} \cdot \mathbf{k} - \nu \mathbf{k} \otimes \mathbf{k}] k^{-2}. \tag{3.18}$$

The dot product of this tensor equation with $\boldsymbol{\varepsilon} \cdot \mathbf{k}$ in both prefactor and postfactor positions eliminates $\boldsymbol{\mu}$ and yields the relation between direct stress and normal displacement

$$Ehc \exp (i\theta) \mathbf{k} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{b} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{k} k^{-2} \eta = \hat{N} \tag{3.19}$$

which gives the midsurface strain coefficient

$$\boldsymbol{\alpha} = \eta \frac{\mathbf{k} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{b} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{k}}{k^4} [\boldsymbol{\varepsilon} \cdot \mathbf{k} \otimes \boldsymbol{\varepsilon} \cdot \mathbf{k} - \nu \mathbf{k} \otimes \mathbf{k}]. \tag{3.20}$$

This may be used in (3.8) to obtain $\mathfrak{L}$ in terms of only η and $\mathbf{k}$:

$$\mathfrak{L} = -\tfrac{1}{2} Ehc^2 \eta^2 F \tag{3.21}$$

where

$$F = F(\mathbf{k}, \mathbf{r}; \omega) = k^4 + \mathbf{k} \cdot \frac{\mathbf{N}_0}{Ehc} \cdot \mathbf{k} + \left(\frac{\mathbf{k} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{b} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{k}}{k^2} \right)^2 - \frac{\rho \omega^2}{E}. \tag{3.22}$$

Now (3.12) gives the recognizable form of the eikonal equation

$$F(\mathbf{k}, \mathbf{r}; \omega) = 0 \tag{3.23}$$

and (3.13) is the transport condition

$$\nabla \cdot (\eta^2 F_{\mathbf{k}}) = 0. \tag{3.24}$$

The explicit calculation of $\boldsymbol{\mu}$ has been avoided in the preceding method. Since $\boldsymbol{\alpha}$ in (3.20) is defined by (3.5), the dot product of (3.20) by $\mathbf{k}$ in both pre- and postfactor positions gives the component in the gradient direction

$$\boldsymbol{\mu} \cdot \mathbf{k} = \eta k^{-2}(\mathbf{k} \cdot \mathbf{b} \cdot \mathbf{k} - \nu \mathbf{k} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{b} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{k}). \tag{3.25}$$

The dot product of (3.20) with $\mathbf{k}$ in the prefactor position and $\boldsymbol{\varepsilon} \cdot \mathbf{k}$ in the postfactor position gives the orthogonal component

$$\mathbf{k} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{\mu} = 2\eta k^{-2} \mathbf{k} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{b} \cdot \mathbf{k}. \tag{3.26}$$

While the shell equations are not at all simple, the use of the time-averaged Lagrangian does provide a clear, economical derivation of the significant relations for the asymptotic analysis. Not obvious is the fact that the relations (3.23) – (3.26) hold even when $\mathbf{k}$ is complex-valued, giving an "evanescent" wave solution (3.1) of exponential rather than sinusoidal spatial variation. In this case, the dependence on θ is not entirely eliminated in the time-averaged Lagrangian, so the Euler equation (3.13) would appear to gain additional terms. However, by the straightforward expansion procedure used by Steele [8], Prat [5] and Logan [3], which is valid for complex $\mathbf{k}$ and for the static case of $\omega \to 0$, the identical relations (3.23) – (3.26) are obtained. This indicates that, instead of using the time-averaged Lagrangian, a more fundamental approach could be taken, in which the possibility of an evanescent wave would be admitted, but which produces the same relations (3.23)–(3.26).

4. Ray coordinates. As is well known, the eikonal equation (3.23) can be solved by the method of characteristics. The characteristic, or ray, emitted from a given point on the surface in a particular direction can be determined from the equations

$$d\mathbf{r}/d\sigma = F_{\mathbf{k}} \tag{4.1}$$

$$(dk/d\sigma) \cdot \boldsymbol{\delta} = -F_{\mathbf{r}} \tag{4.2}$$

while the parameter along the ray σ is related to the phase function ζ by

$$d\zeta/d\sigma = \mathbf{k} \cdot F_{\mathbf{k}} . \tag{4.3}$$

If the rays emitted from a boundary curve are used as one family of coordinate lines, and the lines of constant ζ, i.e. constant phase, are used for the second set of coordinate lines, then we have

$$\begin{aligned} x^1 &= \zeta, \\ \mathbf{a}^1 &= \nabla \zeta = \mathbf{k}, \\ \mathbf{a}_1 &= \partial \mathbf{r}/\partial \zeta = F_{\mathbf{k}}/\mathbf{k} \cdot F_{\mathbf{k}} , \\ \mathbf{a}_2 &= a\mathbf{k} \cdot \boldsymbol{\varepsilon}. \end{aligned} \tag{4.4}$$

Note that the variational problem of finding the curve Γ on the surface which has endpoints A and B, and which extremizes the integral

$$\mu = \int_A^B \frac{d\mathbf{r}}{d\sigma} \cdot \mathbf{k} \, d\sigma \tag{4.5}$$

subject to the constraint condition (3.23) has the Euler equations (4.1) and (4.2). If we identify $\mathbf{k}$ as the gradient of the scalar ζ, then

$$\mu = \int_A^B d\zeta = \zeta(B) - \zeta(A). \tag{4.6}$$

Thus the rays can be thought of as lines of minimum phase change subject to the constraint (3.23). Generally the rays do not coincide with the gradients, so the ray coordinates are not orthogonal.

In terms of the ray coordinates, the amplitude function can be easily obtained. The condition (3.24) is

$$0 = \nabla \cdot (\eta^2 \mathbf{k} \cdot F_{\mathbf{k}} \mathbf{a}_1) = \frac{\partial}{\partial x^1} (\eta^2 \mathbf{k} \cdot F_{\mathbf{k}}) + \eta^2 \mathbf{k} \cdot F_{\mathbf{k}} \frac{\partial \mathbf{a}}{\mathbf{a} \, \partial x^1} . \tag{4.7}$$

Thus

$$\eta = (a \mathbf{k} \cdot F_{\mathbf{k}})^{-1/2}. \tag{4.8}$$

The determinate of the metric a can be obtained from the calculation of the spacing between two adjacent rays. In the special case considered by Steele [9], a linear second-order equation on a single ray was obtained for a. Hayes [2] provides equations for the calculation of a along a single ray in the general case.

5. Variation in properties. The restriction to constant c in Sec. 3 was made so that the relative orders of magnitude of the various terms could be explicitly shown with a minimum of notational clutter. A smooth variation in thickness and in E can, however, be easily taken into consideration. Writing

$$\xi = c^{-1/2}\zeta, \qquad \bar{\mathbf{k}} = \nabla \xi, \tag{5.1}$$

we obtain, instead of (3.21),

$$\mathfrak{L} = -\tfrac{1}{2}\eta^2 G \tag{5.2}$$

where

$$G = G(\bar{\mathbf{k}}, \mathbf{r}, \omega) = Ehc^2 \left\{ c^2(\bar{k})^{-4} + \bar{\mathbf{k}} \cdot \frac{\mathbf{N}_0}{Eh} \cdot \bar{\mathbf{k}} + \left(\frac{\bar{\mathbf{k}} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{b} \cdot \boldsymbol{\varepsilon} \cdot \bar{\mathbf{k}}}{\bar{k}^2} \right)^2 - \frac{\rho \omega^2}{E} \right\} \tag{5.3}$$

and the obvious modifications of the equations in Sec. 4.

6. Transient excitation. For the important problems of shell vibration and steady-state wave propagation the coordinates (4.4) are convenient. For transients, the formulation discussed of Whitham [10] and Hayes [2] is preferred. Instead of (3.2), a more general form of the phase function is considered, with

$$\frac{\partial \theta}{\partial t} = -\omega; \qquad \nabla \theta = \bar{\mathbf{k}}. \tag{6.1}$$

The time-averaged Lagrangian retains the same form (5.2). Explicit time dependence, for instance in the membrane prestress, may now be admitted. The Euler equations are

$$\partial \mathfrak{L}/\partial \eta = 0 \tag{6.2}$$

which gives the same eikonal equation

$$G(\bar{\mathbf{k}}, \mathbf{r}, \omega, t) = 0 \tag{6.3}$$

but the time dependence effects the transport condition

$$-\frac{\partial}{\partial t}\mathfrak{L}_\omega + \nabla\cdot\mathfrak{L}_{\mathbf{k}} = 0. \tag{6.4}$$

The solution of (6.3) on a ray is governed by

$$\begin{gathered} d\mathbf{r}/d\sigma = G_{\mathbf{k}}, \qquad (d\bar{\mathbf{k}}/d\sigma)\cdot\boldsymbol{\delta} = -G_{\mathbf{r}}, \\ dt/d\sigma = -G_\omega, \qquad d\omega/d\sigma = G_t, \qquad d\theta/d\sigma = \bar{\mathbf{k}}\cdot G_{\mathbf{k}} + \omega G_\omega . \end{gathered} \tag{6.5}$$

Time instead of σ can be used as the parameter along the ray. In particular, the local group velocity is

$$d\mathbf{r}/dt = -G_{\mathbf{k}}/G_\omega . \tag{6.6}$$

For a given frequency the ray trajectory on the surface given by (6.5) is the same as that given by (4.1) and (4.2). In the transient situation, the "instantaneous frequency" at a point and the corresponding direction of energy propagation change with time.

7. Vibration and stability. Much of the local behavior of the shell can be obtained from consideration of the eikonal equation (3.23). For a given direction of the gradient $\mathbf{k} = \mathbf{a}^1$, (3.23) is a polynomial for the amplitude $k = |\mathbf{k}|$.

$$k^4 + k^2(N_0^{\langle 11\rangle}/Ehc) + b_{\langle 22\rangle}^{\ 2} - \frac{\rho\omega^2}{E} = 0 \tag{7.1}$$

in which $N_0^{\langle 11\rangle}$ is the physical component of membrane prestress in the direction of $\mathbf{k}$ and $b_{\langle 22\rangle}$ is the curvature of the surface in the orthogonal direction. The roots of (7.1) are

$$k^2 = -(N_0^{\langle 11\rangle}/2Ehc) \pm \left[(N_0^{\langle 11\rangle}/2Ehc)^2 + \frac{\rho\omega^2}{E} - b_{\langle 22\rangle}^{\ 2}\right]^{1/2}. \tag{7.2}$$

When the prestress is tensile or zero in the direction of $\mathbf{k}$, a positive root occurs when

$$\rho\omega^2/E > b_{\langle 22\rangle}^{\ 2}. \tag{7.3}$$

The limit gives the "cut-off" frequency for (bending) wave propagation in the direction $\mathbf{k}$, which is equal to the resonant frequency in pure expansion of the cylindrical and spherical shells, and gives the transition point on the shell of revolution discussed by Ross [7]. When the prestress is compressive, the positive root occurs for frequencies

$$\rho\omega^2/E > b_{\langle 22\rangle}^{\ 2} - (N_0^{\langle 11\rangle}/2Ehc)^2. \tag{7.4}$$

In this case the prestress lowers the cutoff frequency. Finally, for sufficiently high compression

$$|N_0^{\langle 11\rangle}/2Ehc\, b_{\langle 22\rangle}| > 1 \tag{7.5}$$

the solution is oscillatory in the direction **k** for zero frequency. An oscillatory solution generally permits the satisfaction of homogeneous boundary conditions with an arbitrary amplitude. Indeed, (7.5) is exactly the "classical" condition for instability of a cylinder under axial force and the sphere under external pressure.

In the problem of the torispherical end closure of a cylindrical vessel with internal pressure, the circumferential stress in the toroidal segment is compressive. Ranjan and Steele [6] find that all the available experimental data for the incidence of circumferential wrinkling correspond to values of the parameter in (7.5) ranging from about 0.9 to 2. Considering the difficulties associated with shell instability, and the oversimplified nature of the local criterion (7.5), such agreement is remarkable.

We conclude that the general approach which has been discussed in this paper offers more than unification and simplification in the derivation of well-known results. Insight and useful results can be obtained for nontrivial problems of difficult geometry.

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