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ON THE STABILITY OF AN OPERATOR EQUATION MODELING NUCLEAR REACTORS WITH DELAYED NEUTRONS*

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1. Introduction. A number of studies [2, 6, 7, 8, 9, 10, 11, 12, 15] have been made of a system of integrodifferential equations which arise as dynamic models of one-dimensional continuous-medium nuclear reactors. In particular, the effect of delayed neutrons is considered in [1, 9].

In this paper we continue to pursue the approach exploited in [4], but consider the effect of delayed neutrons which was omitted in that paper. In particular, we wish to consider the system of equations

$$\frac{d}{dt}p(t) = -p(t)\int_\Omega a(z)T(t,z)\,dz - \frac{\beta}{l^*}p(t) + \sum_{i=1}^m \lambda_i c_i(t),$$

$$\frac{d}{dt}c_i(t) = \frac{\beta_i}{l^*}p(t) - \lambda_i c_i(t), \qquad i = 1, 2, \cdots, m. \tag{1.1}$$

$$\frac{\partial}{\partial t}T(t,z) = e(z)[p(t) - \nu] + \frac{\partial^2}{\partial z^2}T(t,z), \qquad z \in \Omega,$$

for $t \geq 0$, with appropriate boundary conditions for T on the boundary $\partial\Omega$ of Ω, a subset of the real line R, and with initial conditions

$$p(0) = p_0, \quad c_i(0) = c_{i0}, \quad i = 1, 2, \cdots, m, \quad T(0,z) = f(z), \quad z \in \Omega. \tag{1.2}$$

This system [9] is an appropriate model for the dynamic behavior of a continuous-medium nuclear reactor with m groups of delayed neutrons with concentrations $c_i(t) \geq 0$. Here, $T(t, z)$ is the deviation of the temperature from equilibrium and $p(t) \geq 0$ is the instantaneous power, with $p(t) \equiv \nu > 0$ at equilibrium. The constants l^*, β_i, λ_i are positive and $\beta = \sum_{i=1}^m \beta_i$. It is noted that the equilibrium $(p(t), c_i(t), T(t,z)) \equiv (\nu, (\beta_i\nu/\lambda_i l^*), 0)$ represents a steady state of operation of the reactor. We refer the interested reader to [1, 9] for a detailed explication of this model.

This system of equations and its variants have been studied extensively by Levin and Nohel [6, 7, 8, 9, 12], Miller [10, 11] and their students [2]. The majority of these studies have viewed this problem as a nonlinear Volterra integrodifferential equation. Here, as in [8, 4], the theory of C_0-semigroups is applied. We consider an abstract equation of which (1.1)–(1.2) is a special case and show that, under appropriate assumptions,

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it generates a C_0-semigroup on a Hilbert space; we then apply appropriate Liapunov direct method arguments to study the asymptotic behavior of the solutions.

Here we extend the results obtained in [4] by including the effect of the delayed neutrons. Our results are similar to those obtained in [9] and are applicable to a more general type of equation, which includes cases in which heat conduction is not necessarily linear; our method of proof is quite different.

Finally, existence, uniqueness and stability results can be obtained, as is shown in what follows, in a variety of norms, depending on the type of result desired.

2. The abstract equation. For our purpose it is convenient to view (1.1) in a slightly different form. Upon defining

$$\begin{aligned} p(t) &= \nu \exp(u(t)), \\ c_i(t) &= \mu_i \exp(q_i(t)), \qquad i = 1, 2, \cdots, m, \\ \mu_i &= \beta_i \nu/\lambda_i l^*, \qquad i = 1, 2, \cdots, m, \end{aligned} \tag{2.1}$$

Eqs. (1.1) become

$$\begin{aligned} \frac{d}{dt} u(t) &= -\int_\Omega a(z)T(t, z)\, dz + \sum_{i=1}^{m} \frac{\beta_i}{l^*} [\exp(q_i(t) - u(t)) - 1], \\ \frac{d}{dt} q_i(t) &= \lambda_i [\exp(u(t) - q_i(t)) - 1], \qquad i = 1, 2, \cdots, m, \\ \frac{\partial}{\partial t} T(t, z) &= \nu e(z)[\exp(u(t)) - 1] + \frac{\partial^2}{\partial z^2} T(t, z), \qquad z \in \Omega, \end{aligned} \tag{2.2}$$

for $t \geq 0$, with appropriate initial and boundary conditions.

Interpreting the integral in (2.2) as the real inner product for $\mathcal{L}_2(\Omega)$, let $\mathcal{H}$ be a real Hilbert space such that $\mathcal{H} \subset \mathcal{L}_2(\Omega)$ and $\|w\|_{\mathcal{L}_2} \leq M \|w\|_{\mathcal{H}}$ for some $M > 0$ and every $w \in \mathcal{H}$. Also, let $L : \mathfrak{D}(L) \to \mathcal{H}$ be an operator, not necessarily linear, such that

i) the domain $\mathfrak{D}(L)$ is dense in $\mathcal{H}$,
ii) the range $R(I + \mu L) = \mathcal{H}$ for all sufficiently small $\mu > 0$, (2.3)
iii) $\langle w_1 - w_2, Lw_1 - Lw_2 \rangle_{\mathcal{H}} \geq -\alpha \|w_1 - w_2\|_{\mathcal{H}}^2$ for every $w_1, w_2 \in \mathfrak{D}(L)$ and some real number α.

Assuming $a \in \mathcal{L}_2(\Omega)$, $e \in \mathcal{H}$, consider for $u : R^+ \to R$, $q_i : R^+ \to R$, $w : R^+ \to \mathcal{H}$, the operator equation

$$\begin{aligned} \dot{u} &= -\langle a, w(t) \rangle_{\mathcal{L}_2} + \sum_{i=1}^{m} \frac{\beta_i}{l^*} [\exp(q_i(t) - u(t)) - 1] \\ \dot{q}_i(t) &= \lambda_i [\exp(u(t) - q_i(t)) - 1], \qquad i = 1, 2, \cdots, m, \\ \dot{w}(t) &= \nu [\exp(u(t)) - 1] e - Lw(t), \end{aligned} \tag{2.4}$$

for $t \geq 0$, with appropriate initial conditions in $R \times R^m \times \mathcal{H}$.

It is clear that (2.4) is a generalization of (2.2) by letting $Lw(z) = -(\partial^2/\partial z^2) w(z)$ and choosing $\mathcal{H}$ as one of the Sobolev spaces $\mathcal{H}_2^n(\Omega)$, $n = 0, 1, \cdots$, with $\mathcal{H}_2^0(\Omega) = \mathcal{L}_2(\Omega)$; in this case (2.3) holds for some $\alpha \leq 0$. In general, (2.3) implies that $0 \in \mathfrak{D}(L)$ and $L0 = 0$; therefore, $(u(t), q_1(t), \cdots, q_m(t), w(t)) = (0, 0, \cdots, 0, 0) \in R \times R^m \times \mathcal{H}$ is an equilibrium state for the system (2.4).

Define $\sigma : R \to R$, $\sigma(0) = 0$, and consider

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0 \in \mathfrak{D}(A) \subset \mathfrak{X}; \tag{2.5}$$

where $\mathfrak{X} = R \times R^m \times \mathcal{H}$ is endowed with inner product

$$\langle x_1 , x_2 \rangle_{\mathfrak{X}} \equiv u_1 u_2 + \sum_{i=1}^{m} q_{i1} q_{i2} + \langle w_1 , w_2 \rangle_{\mathcal{H}} \quad \text{for} \quad x_j = (u_j , q_{1j} , \cdots , q_{mj} , w_j) \in \mathfrak{X},$$

$$j = 1, 2; \; \mathfrak{D}(A) = \{(u, q_1 , \cdots , q_m , w) \in \mathfrak{X} \mid w \in \mathfrak{D}(L)\};$$

and

$$Ax = \left(-\langle a, w \rangle_{\mathfrak{L}_2} + \sum_{i=1}^{m} \frac{\beta_i}{l^*} \sigma(q_i - u), \lambda_1 \sigma(u - q_1), \cdots , \lambda_m \sigma(u - q_m), \nu\sigma(u)e - Lw \right). \tag{2.6}$$

We remark that (2.5) includes (2.4) (in this case $\sigma(z) = \exp(z) - 1$, $z \in R$), and that $A0 = 0$, which implies that $x = 0$ is an equilibrium solution of (2.5).

We now show, under a restriction not satisfied by the nonlinearity σ of interest to us, that (2.5) generates a C_0-semigroup.

Proposition 1: Let L satisfy (2.3) and assume that σ is uniformly Lipschitz continuous on R. Then

$$S(t)x = \lim_{n \to \infty} \left(I - \frac{t}{n} A \right)^{-n} x, \qquad x \in \mathfrak{X}, \qquad t \geq 0, \tag{2.7}$$

defines a C_0-semigroup $\{S(t)\}_{t \geq 0}$ on $\mathfrak{X}$; moreover, if $x_0 \in \mathfrak{D}(A)$, then $S(t)x_0$ is the unique strong solution of (2.5).

Proof: The assumptions on L imply that $\mathfrak{D}(A)$ is dense in $\mathfrak{X}$. Let K be the Lipschitz constant associated with σ. Then, for any $x_1 , x_2 \in \mathfrak{D}(A)$ we have that

$$\begin{aligned}\langle x_1 - x_2 , Ax_1 - Ax_2 \rangle_{\mathfrak{X}} &\leq (M \, ||a||_{\mathfrak{L}_2} + \nu K \, ||e||_{\mathcal{H}}) \, |u_1 - u_2| \, ||w_1 - w_2||_{\mathcal{H}} \\ &\quad + \frac{K\beta}{l^*} |u_1 - u_2|^2 + K \, |u_1 - u_2| \sum_{i=1}^{m} \left(\frac{\beta_i}{l^*} + \lambda_i \right) |q_{i1} - q_{i2}| \\ &\quad + K \sum_{i=1}^{m} \lambda_i \, |q_{i1} - q_{i2}|^2 + \alpha \, ||w_1 + w_2||_{\mathcal{H}}^2 \\ &\leq \omega \, ||x_1 - x_2||_{\mathfrak{X}}^2\end{aligned}$$

for some real number ω. Hence, $\omega I - A$ is accretive [3].

We claim that $\mathfrak{R}(I - \mu A) = \mathfrak{X}$ for some sufficiently small $\mu > 0$. Indeed, it follows from (2.3) that for every positive μ such that $\mu\alpha < 1$ the operator $I + \mu L$ is invertible on $\mathcal{H}$ and $||(I + \mu L)^{-1} w||_{\mathcal{H}} \leq (1 - \mu\alpha)^{-1} ||w||_{\mathcal{H}}$. If we define $H_{\mu,i} : R \to R$ by $H_{\mu,i} z = z + \mu\lambda_i \sigma(z)$, $i = 1, 2, \cdots , m$, the uniform Lipschitz condition on σ implies that $H_{\mu,i}$ is invertible on R for $0 < \mu < 1/K\lambda_i$ and $|H_{\mu,i}^{-1} z| \leq (1 - \mu K \lambda_i)^{-1} |z|$. Letting $(\hat{q}_1 , \cdots , \hat{q}_m , \hat{w})$ be a fixed but arbitrary element of $R^m \times \mathcal{H}$ and defining $F_\mu : R \to R$ by

$$F_\mu(u) = u + \mu \langle a, (I + \mu L)^{-1} (\hat{w} + \mu\nu\sigma(u)e) \rangle_{\mathfrak{L}_2} - \mu \sum_{i=1}^{m} \frac{\beta_i}{l^*} \sigma(-H_{\mu,i}^{-1}(u - \hat{q}_i)),$$

it follows that, for every sufficiently small $\mu > 0$, F_μ is continuous and $F_\mu(u) \to \infty \; (-\infty)$

as $u \to \infty$ $(-\infty)$. Hence, $\mathcal{R}(F_\mu) = R$ for every sufficiently small $\mu > 0$; in turn, this implies that $\mathcal{R}(I - \mu A) = \mathfrak{X}$ for every sufficiently small $\mu > 0$.

Application of Theorems I and II of [3] imply that (2, 7) generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$, and, if $x(\cdot) : R^+ \to \mathfrak{X}$ is a strong solution of (2.5), then $x(t) = S(t)x(0)$, $t \geq 0$. Furthermore, by Theorem 7.1 of [5] there exists a unique solution of (2.5) for every $x_0 \in \mathfrak{D}(A)$. The proof is complete.

We remark that in [3] it is also shown that $\{S(t)\}_{t\geq 0}$ is of class $Q_\omega(\mathfrak{X})$, i.e.,

$$||S(t)x_1 - S(t)x_2||_{\mathfrak{X}} \leq ||x_1 - x_2||_{\mathfrak{X}} \exp(\omega t), \qquad t \geq 0,$$

for every $x_1, x_2 \in \mathfrak{X}$. Also, Theorem 7.5 of [5] assures us that, for $x \in \mathfrak{D}(A)$, $AS(\cdot)x$ is right continuous on R^+, $S(\cdot)x$ is right differentiable on R^+, and this right derivative equals $AS(t)x$ for every $t \in R^+$.

We now proceed to use these results to show, through a stability argument, that we can relax the conditions on σ so as to encompass our exponential nonlinearity.

3. The main results. We now present results for (2.5)–(2.6) which are applicable to (2.4).

Proposition 2: Let L satisfy (2.3) and assume the following conditions hold:

1. $\sigma : R \to R$ is locally Lipschitz continuous on R and
   i) for every $\delta > 0$ there exists $\eta_\delta > 0$ such that $u\sigma(u) \geq \eta_\delta u^2$ for every $u \in R$ with $|u| \leq \delta$,
   ii) $\displaystyle\int_0^u \sigma(z)\,dz \to \infty \quad \text{as} \quad |u| \to \infty,$
   iii) for every $u_1, u_2 \in R$, $\sigma(u_1)\sigma(u_2 - u_1) + \sigma(u_2)\sigma(u_1 - u_2) \leq 0$.
2. There exists a symmetric bounded linear operator $G : \mathcal{H} \to \mathcal{H}$ such that
   i) $\langle w, Gw\rangle_{\mathcal{H}} \geq \gamma\, ||w||_{\mathcal{H}}^2$ for some $\gamma > 0$ and every $w \in \mathcal{H}$,
   ii) $\langle Ge, w\rangle_{\mathcal{H}} = \langle a, w\rangle_{\mathcal{L}_2}$ for every $w \in \mathcal{H}$,
   iii) $\langle w, GLw\rangle_{\mathcal{H}} \geq 0$ for every $w \in \mathfrak{D}(L)$.

Then (2.6)–(2.7) define a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on $\mathfrak{X}$ and $S(\cdot)x : R^+ \to \mathfrak{X}$ is the unique strong solution of (2.5) for every $x \in \mathfrak{D}(A)$; moreover, the equilibrium $x = 0$ is stable and, for each $\rho > 0$, the set $\mathcal{C}_\rho \equiv \{x \in \mathfrak{X} \mid V(x) \leq \rho\}$ is bounded and positively invariant under $\{S(t)\}_{t\geq 0}$, where

$$V(x) = \nu \int_0^u \sigma(z)\,dz + \sum_{i=1}^m \mu_i \int_0^{q_i} \sigma(z)\,dz + \tfrac{1}{2}\langle w, Gw\rangle_{\mathcal{H}}$$

for $(u, q_1, \cdots, q_m, w) = x \in \mathfrak{X}$.

Proof: The assumptions on σ imply that each $\mathcal{C}_\rho$ is bounded, $\mathcal{C}_{\rho_1} \subset \mathcal{C}_{\rho_2}$ for $\rho_1 < \rho_2$, and $\mathfrak{X} = \bigcup_{\rho>0} \mathcal{C}_\rho$. Consider a fixed but arbitrary $\rho > 0$ and the corresponding $\mathcal{C}_\rho$. Since $\mathcal{C}_\rho$ is bounded, it follows that for $x \in \mathcal{C}_\rho$ the function σ can be identified with some uniformly Lipshitz continuous function $\sigma_\rho : R \to R$. If we denote by A_ρ the operator defined by (2.6) on $\mathfrak{D}(A_\rho) = \mathfrak{D}(A)$ with σ replaced by σ_ρ, it follows from Proposition 1 that A_ρ generates a C_0-semigroup $\{S_\rho(t)\}_{t\geq 0}$ on $\mathfrak{X}$. Since $S_\rho(\cdot)x$ is everywhere right differentiable on R^+ for $x \in \mathfrak{D}(A)$, it follows that the functional

$$\dot{V}(x) \equiv \overline{\lim_{t\searrow 0}}\, \frac{1}{t}\,[V(S_\rho(t)x) - V(x)], \qquad x \in \mathcal{C}_\rho,$$

is given by

$$\dot{V}(x) = -\langle w, GLw\rangle_{\mathcal{H}} + \sum_{i=1}^{m} \mu_i \lambda_i [\sigma(u)\sigma(q_i - u) + \sigma(q_i)\sigma(u - q_i)]$$

$$\leq 0$$

for $(u, q_1, \cdots, q_m, w) = x \in \mathfrak{D}(A) \cap \mathcal{C}_\rho$; hence, the denseness of $\mathfrak{D}(A)$ and the fact that $\{S_\rho(t)\}_{t\geq 0}$ is of class $Q_{\omega_\rho}(\mathfrak{X})$, together with the fact that $V(S_\rho(\cdot)x)$ is nonincreasing for $x \in \mathfrak{D}(A) \cap \mathcal{C}_\rho$, imply that $V(S_\rho(\cdot)x)$ is nonincreasing for $x \in \mathcal{C}_\rho$. Therefore $\mathcal{C}_\rho$ is positively invariant under $\{S_\rho(t)\}_{t\geq 0}$.

The assumptions of the theorem also imply the existence of positive numbers α_1 , α_2 such that $\alpha_1 ||x||^2 \leq V(x) \leq \alpha_2 ||x||^2$ for every $x \in \mathcal{C}_\rho$; hence $x = 0$ is a stable equilibrium of $\{S_\rho(t)\}_{t\geq 0}$.

Since $Ax = A_\rho x$ on $\mathfrak{D}(A) \cap \mathcal{C}_\rho$ and $\mathfrak{X} = \bigcup_{\rho>0} \mathcal{C}_\rho$, it follows that A generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on $\mathfrak{X}$ defined by (2.7); moreover, $S(t)x = S_\rho(t)x$ for $x \in \mathcal{C}_\rho$, $t \geq 0$. Therefore $\{S(t)\}_{t\geq 0}$ has a stable equilibrium at $x = 0$, $S(\cdot)x : R^+ \to \mathfrak{X}$ is the unique strong solution of (2.5) for every $x \in \mathfrak{D}(A)$, and each $\mathcal{C}_\rho$ is positively invariant under $\{S(t)\}_{t\geq 0}$. This concludes the proof.

At this juncture, some remarks seem appropriate. First of all, from the proof of this proposition and the remarks after Proposition 1, it follows that for $x \in \mathfrak{D}(A)$, $S(\cdot)x$ is continuously right differentiable on R^+ and its right derivative is $AS(\cdot)x$. Secondly, for each $\rho > 0$, it is seen that $\{S(t)\}_{t\geq 0}$ is of class $Q_{\omega_\rho}(\mathcal{C}_\rho)$, where it is noted that ω_ρ might depend upon ρ.

Finally, condition 1(iii) of Proposition 2 is needed because of the particular Liapunov functional used in the proof. We note that 1(iii) is satisfied by any odd monotonic function; it is also satisfied by $\sigma(z) = \exp(z) - 1$, the function of particular interest here.

We now proceed to strengthen our stability result by adding some further conditions to those of Proposition 2.

THEOREM: Let σ be given by $\sigma(z) = e^z - 1$ for $z \in \mathfrak{R}$, let L satisfying (2.3) be such that $||CLw||_{\mathcal{H}}/||w||_{\mathcal{H}}$ is bounded on bounded subsets of $\mathfrak{D}(L)$ for some bounded linear operator $C : \mathcal{H} \to \mathcal{H}$. Assume that condition (2) of Proposition 2 is satisfied and, for every $\delta > 0$, there exists a $\mu_\delta \geq 0$ such that $\langle w, GLw\rangle_{\mathcal{H}} \geq \mu_\delta ||w||_{\mathcal{H}}^2$ for every $w \in \mathcal{H}$ with $||w||_{\mathcal{H}} \leq \delta$. Then (2.6)–(2.7) define a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on $\mathfrak{X}$ and $S(\cdot)x : R^+ \to \mathfrak{X}$ is the unique strong solution of (2.5) for every $x \in \mathfrak{D}(A)$; moreover, the equilibrium $x = 0$ is stable and, for each $\rho > 0$, the set $\mathcal{C}_\rho \equiv \{x \in \mathfrak{X} \mid V(x) \leq \rho\}$ is bounded and positively invariant under $\{S(t)\}_{t\geq 0}$, where

$$V(x) \equiv \nu \int_0^u \sigma(z)\, dz + \sum_{i=1}^{m} \mu_i \int_0^{q_i} \sigma(z)\, dz + \tfrac{1}{2}\langle w, Gw\rangle_{\mathcal{H}}$$

for $(u, q_1, \cdots, q_m, w) = x \in \mathfrak{X}$. If $||Ce||_{\mathcal{H}} \neq 0$ and $\mu_\delta > 0$ for some $\delta > 0$, the equilibrium is exponentially asymptotically stable; if $||Ce||_{\mathcal{H}} \neq 0$ and $\mu_\delta > 0$ for every $\delta > 0$, then to each bounded set $\mathfrak{B} \subset \mathfrak{X}$ there corresponds positive numbers $\bar{M}$ and $\bar{\omega}$ such that

$$||S(t)x||_{\mathfrak{X}} \leq \bar{M}\, ||x||_{\mathfrak{X}} \exp(-\bar{\omega}t)$$

for every $x \in \mathfrak{B}$, $t \geq 0$.

Proof: Since all assumptions of Proposition 2 are satisfied, only the conclusions pertaining to the assumption $||Ce||_{\mathcal{H}} \neq 0$ remain to be proved. For $||Ce||_{\mathcal{H}} \neq 0$ and

$\mu_\delta > 0$ for some $\delta > 0$, choose $\rho = \gamma\delta^2$ and note that

$$\dot{V} \leq -\mu_\delta \, ||w||_{\mathcal{H}}^2 - \sum_{i=1}^{m} \mu_i\lambda_i(q_i - u)^2$$

for every $(u, q_1, \cdots, q_m, w) = x \in \mathfrak{D}(A) \cap \mathfrak{C}_\rho$.

Define a function $U(x) \equiv -2u\langle Ce, Cw\rangle_{\mathcal{H}}$ where $(u, q_1, \cdots, q_m, w) = x \in \mathfrak{X}$; then, for $x \in \mathfrak{D}(A) \cap \mathfrak{C}_\rho$, it follows that

$$\begin{aligned}
\dot{U}(x) &\equiv \overline{\lim_{t\to 0}} \frac{1}{t} [U(S(t)x - U(x)] \\
&= -2\langle Ce, Cw\rangle_{\mathcal{H}}\left[-\langle a, w\rangle_{\mathcal{L}_2} + \sum_{i=1}^{m} \frac{\beta_i}{l^*} (\exp(q_i - u) - 1)\right] \\
&\qquad\qquad - 2\nu u(\exp(u) - 1) \, ||Ce||_{\mathcal{H}}^2 + 2u\langle Ce, CLw\rangle_{\mathcal{H}} \\
&\leq -2\nu K_1 u^2 + 2 \, ||Ce||_{\mathcal{H}} \, ||C|| \left(M \, ||a||_{\mathcal{L}_2} \, ||w||_{\mathcal{H}}^2 + K_2 \, ||w||_{\mathcal{H}} \sum_{i=1}^{m} \frac{\beta_i}{l^*} |q_i - u|\right) \\
&\qquad\qquad + 2K_3 \, ||Ce||_{\mathcal{H}} \, |u| \, ||w||_{\mathcal{H}}
\end{aligned}$$

for some positive numbers K_1, K_2, K_3, depending on ρ. Since $\nu > 0$, $\mu_i\lambda_i > 0$, and $\mu_\delta > 0$, it follows that there exists a sufficiently small $\theta > 0$ such that

$$\begin{aligned}
\alpha_3 \, ||x||_{\mathfrak{X}}^2 \leq V(x) + \theta U(x) &\leq \alpha_4 \, ||x||_{\mathfrak{X}}^2, \qquad x \in \mathfrak{X}, \\
\dot{V}(x) + \theta \dot{U}(x) &\leq -\alpha_5 \, ||x||_{\mathfrak{X}}^2, \qquad x \in \mathfrak{D}(A) \cap \mathfrak{C}_\rho,
\end{aligned}$$

for some positive numbers α_3, α_4, α_5 depending only on $\rho = \gamma\delta^2$; moreover, the denseness of $\mathfrak{D}(A)$ and the fact that $\{S(t)\}_{t\geq 0}$ is of class Q_{ω_ρ} imply that the last inequality holds for every $x \in \mathfrak{C}_\rho$. It now follows that

$$||S(t)x||_{\mathfrak{X}}^2 \leq (\alpha_4/\alpha_3) \, ||x||_{\mathfrak{X}}^2 \exp(-(\alpha_5/\alpha_4)t), \qquad x \in \mathfrak{C}_\rho ;$$

hence, the equilibrium $x = 0$ is exponentially asymptotically stable. Since $\bigcup_{\rho>0} \mathfrak{C}_\rho = \mathfrak{X}$, each bounded set $\mathfrak{B} \subset \mathfrak{X}$ is contained in some $\mathfrak{C}_\rho$, $\rho > 0$; hence; the proof is complete.

If L is linear, the existence of a suitable operator C is assured, since (2.3) implies that $L + \beta I$ has a bounded inverse for all sufficiently large $\beta > 0$; in this case we may choose $C \equiv (L + \beta I)^{-1}$ and note that $||e||_{\mathcal{H}} \neq 0$ implies $||Ce||_{\mathcal{H}} \neq 0$.

4. Some remarks. First of all, it should be remarked that our stability theorems imply that if the effect of delayed neutrons is destabilizing, this destabilizing effect is sufficiently weak that we cannot detect it through the use of our Liapunov functional $V(x)$; that is, if the system (1.1) with $m = 0$ satisfies our condition for stability, so does the system (1.1) with any positive m. This result is not surprising, since our conditions for stability are only sufficient.

Secondly, our stability results above depend on the existence of the bounded, linear, symmetric operator $G : \mathcal{H} \to \mathcal{H}$ that satisfies two conditions: (i) $\langle Ge, w\rangle_{\mathcal{H}} = \langle a, w\rangle_{\mathcal{L}_2}$ for every $w \in \mathcal{H}$, and (ii) $\langle w, Gw\rangle$ is a Liapunov functional which proves stability (possibly asymptotic stability) for the abstract equation $\dot{w} = Lw$, $w \in \mathfrak{D}(L) \subset \mathcal{H}$. Of course, the determination of such a G depends on the specific operator L and its properties. In [4], where delayed neutrons were ignored and $\mathcal{H}$ was chosen as $\mathcal{L}_2(\Omega)$ with $L : \mathfrak{D}(L) \to$

$\mathcal{L}_2(\Omega)$ a linear nonnegative symmetric operator, we required an operator G that satisfied conditions identical to those stated here. For L a linear second-order differential operator, such as appears in (1.1), we displayed several possible forms for G which yielded a class of stability conditions. These same forms for G are applicable to the study of (1.1). We refer the interested reader to [4].

Finally, as well as incorporating the effects of delayed neutrons, we have improved upon the results of [4] in two directions. Since L is permitted to be nonlinear, generalizations of (1.1) which involve the effects of nonlinear heat conduction can be studied; secondly, due to the somewhat free choice for $\mathcal{H} \subset \mathcal{L}_2(\Omega)$, existence, uniqueness, and stability results can be obtained for (1.1) in terms of a variety of norms.

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