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STABILITY UNDER STEP LOADING OF INFINITELY LONG COLUMNS WITH LOCALIZED IMPERFECTIONS*

By

JOHN C. AMAZIGO
Rensselaer Polytechnic Institute

AND

DEBORAH F. LOCKHART
S.U.N.Y., Geneseo

Abstract. The dynamic buckling of a long column with small dimple imperfections resting on a nonlinear foundation and subjected to axial step-loading is studied using a formal multi-variable perturbation expansion. Simple asymptotic formulas are obtained for the dynamic buckling load and lateral deflection in terms of the Fourier transform of the imperfection. It is found that the static and dynamic buckling loads are equal.

Introduction. The existence of small geometrical and physical imperfections in certain structures leads to large reductions in their buckling strengths. Such structures are known as "imperfection-sensitive". The first general static theory of the post-buckling behavior of these structures is the well-known theory of Koiter [1]. Budiansky and Hutchinson [2, 3, 4] have extended this theory to dynamic buckling. These theories are based essentially on the assumption that the imperfections are in the shape of the classical buckling mode. In [5] the authors showed that this restrictive assumption need not be made to obtain asymptotic expressions for dynamic buckling loads. Here we consider an infinitely long column with an initial imperfection in the shape of a localized dimple. The column rests on a nonlinear elastic foundation and is subjected to an axial load. The static problem has been studied in [6] using equivalent linearization as well as a perturbation expansion involving double scaling in the spatial variable. If the initial imperfection is small, these two methods yield the same expression for the static buckling load in terms of the amplitude of the imperfection. We consider the extension of these results to time-dependent loadings. In this paper we present the case of suddenly applied loads that are subsequently maintained at a constant value.

Differential equation. We consider an infinitely long column with a small localized initial imperfection resting on a nonlinear foundation which is subjected to an axial compressive load. The load is suddenly applied and thereafter maintained at a constant value. The nondimensional form of the equation for the lateral displacement $w(x, t)$ of the column is

$$w_{tt} + w_{xxxx} + 2\lambda w_{xx} + w - \alpha w^3 = -2\lambda \epsilon w_{0xx} \tag{1}$$

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where an alphabetic subscript denotes partial derivative, and the nondimensional axial coordinate x, lateral displacement w, axial load parameter λ, stress-free initial displacement w_0 and time t are related to the corresponding physical quantities by

$$x = (k_1/EI)^{1/4}X, \quad w = (k_3/k_1)^{1/2}W, \quad \lambda = P/2(EIk_1)^{1/2}$$
$$\epsilon w_0 = (k_3/k_1)^{1/2}W_0, \qquad t = (k_1/m)^{1/2}T.$$

Axial inertia and nonlinear geometric effects are neglected. The assumption is made that the initial displacements and velocities are zero. As shown in [5], this is equivalent to assuming that the nondimensional initial displacements and velocities are of the same order as the imperfections. ϵ is a small imperfection parameter, EI is the bending stiffness of the column, P is the magnitude of the axial step loading applied at time $T = 0$, and m is the mass per unit length of the column. The column is restrained against additional lateral deflection W by a foundation that produces a restoring force per unit length $k_1W - \alpha k_3W^3$. α takes on the value 1 or -1 depending on whether the foundation behaves like a "softening" or "hardening" spring. We assume that the imperfection shape $w_0(x)$ is continuously differentiable and satisfies the exponential decay condition

$$|w(x)| < M \exp(-\beta |x|)(M, \beta > 0). \tag{2}$$

The classical theory (linear, time-independent eigenvalue problem with $w_0 \equiv 0$) for any length column with simply supported ends consists of

$$w_{xxxx} + 2\lambda w_{xx} + w = 0, \qquad w = w_{xx} = 0 \quad \text{at} \quad x = 0, r\pi,$$

where r is a measure of the length of the column. The eigenfunctions are

$$w_n(x) = \sin(nx/r), \qquad n = 1, 2, \cdots$$

with corresponding eigenvalues

$$\lambda_n = \tfrac{1}{2}(r/n)^2[1 + (n/r)^4], \qquad n = 1, 2, \cdots.$$

For columns of length given by r an integer, the classical buckling load (lowest eigenvalue) is $\lambda = 1$, corresponding to $n = r$. If r is not an integer, the buckling load λ_c satisfies the inequality

$$0 < \lambda_c - 1 < 1/[2n(n + 1)], \qquad n < r < n + 1.$$

Thus, for $r \gg 1$, $\lambda_c = 1 + 0(r^{-2})$, we consider the column to be of infinite length. The case for which $r = 1$ and the imperfection is arbitrary has been discussed in [5].

Dynamic theory. The problem to be considered is

$$w_{tt} + w_{xxxx} + 2\lambda w_{xx} + w - \alpha w^3 = -2\lambda\epsilon w_{0xx}, \quad -\infty < x < \infty, \quad t > 0, \tag{3}$$

$$\begin{gathered} w, w_x \to 0 \quad \text{as} \quad x \to \pm\infty, \\ w = w_t = 0 \quad \text{at} \quad t = 0. \end{gathered} \tag{4}$$

We consider $\epsilon \ll 1$ and seek to determine the maximum value λ_D of λ less than the classical buckling load λ_c such that the deflection w vanishes as $|x| \to \infty$. This condition must be distinguished from the boundedness condition imposed for imperfections in the shape of classical buckling modes (see [4]) and for finite-length structures (see [5]). This condition was imposed in the solution of the time-independent problem [6]; however, it was not specifically noted in that report. As in [6], a perturbation parameter δ may be introduced and defined as

$$\delta^2 = 2(1 - \lambda). \tag{5}$$

As in this problem, we introduce a new variable $\zeta \equiv \delta x$. As shown in the finite column problem [5], the dominant response depends only on the long-time scale $\tau = \delta t$ and not on the short-time scale t. We assume that w is a function of x, ζ, and τ, write $w(x, t) \equiv u(x, \zeta, \tau; \delta)$, and expand w and $\lambda\epsilon$ in power series in δ:

$$w(x, t) \equiv u(x, \zeta, \tau; \delta) = \sum_{n=1}^{\infty} u^{(n)}(x, \zeta, \tau)\delta^n, \tag{6}$$

$$\lambda\epsilon = \sum_{n=1}^{\infty} \epsilon^{(n)} \delta^n. \tag{7}$$

Substituting these expansions into (3) and equating like powers of δ leads to the sequence of equations:

$$Lu^{(1)} = -2\epsilon^{(1)} w_{0xx}, \tag{8}$$

$$Lu^{(2)} = -2\epsilon^{(2)} w_{0xx} - 4u_{xxx\zeta}{}^{(1)} - 4u_{x\zeta}{}^{(1)}, \tag{9}$$

$$Lu^{(3)} = -2\epsilon^{(3)} w_{0xx} - 4u_{xxx\zeta}{}^{(2)} - 4u_{x\zeta}{}^{(2)} - u_{\tau\tau}{}^{(1)}$$
$$-6u_{xx\zeta\zeta}{}^{(1)} - 2u_{\zeta\zeta}{}^{(1)} + u_{xx}{}^{(1)} + \alpha(u^{(1)})^3, \qquad \text{etc.}, \tag{10}$$

where $L \equiv (\partial^2/\partial x^2 + 1)^2$. The initial conditions (4) become

$$u^{(n)}(x, \zeta, 0) = u_\tau{}^{(n)}(x, \zeta, 0) = 0, \qquad n = 1, 2, \cdots. \tag{11}$$

Guided by the analysis in [6], we admit the possibility of discontinuities in the $u^{(n)}$s or their derivatives at $x = 0$ and $\zeta = 0$. However, we insist that w, w_x, w_{xx}, and w_{xxx} *must* be continuous since they correspond to displacement, slope, moment, and shear respectively. When these continuity conditions are applied to the expansion (6), it is found that the following combinations of functions must be continuous in x and ζ:

$$u^{(n)}, \tag{12}$$

$$u_x{}^{(n)} + u_\zeta{}^{(n-1)}, \tag{13}$$

$$u_{xx}{}^{(n)} + 2u_{x\zeta}{}^{(n-1)} + u_{\zeta\zeta}{}^{(n-2)}, \tag{14}$$

$$u_{xxx}{}^{(n)} + 3u_{xx\zeta}{}^{(n-1)} + 3u_{x\zeta\zeta}{}^{(n-2)} + u_{\zeta\zeta\zeta}{}^{(n-3)}, \tag{15}$$

for $n = 1, 2, 3, \cdots$, where $u^{(k)} \equiv 0$ for $k \leq 0$.

The real-valued bounded solution to (8) is

$$u^{(1)}(x, \zeta, \tau) = a^{(1)}(\zeta, \tau) \exp(ix) + \bar{a}^{(1)}(\zeta, \tau) \exp(-ix) + f^{(1)}(x) \tag{16}$$

where $(\bar{\ })$ denotes complex conjugate of ( ), and $f^{(1)}(x)$ is a particular solution of

$$Lf^{(n)} = -2\epsilon^{(n)} w_{0xx}, \quad \text{for} \quad n = 1. \tag{17}$$

We stipulate that the $f^{(n)}$'s have bounded Fourier transforms $\bar{f}^{(n)}(\omega)$, but may have jumps at $x = 0$ and $\zeta = 0$. Let

$$[f^{(n)}] \equiv f^{(n)}(0^+) - f^{(n)}(0^-), \qquad [f^{(n)\prime}] \equiv f^{(n)\prime}(0^+) - f^{(n)\prime}(0^-), \qquad \text{etc.,}$$

where prime $'$ denotes differentiation with respect to the argument of the function. Thus with $n = 1$

$$\tilde{f}^{(n)}(\omega) = H^{(n)}(\omega)/(\omega^2 - 1)^2 \tag{18}$$

where

$$H^{(n)}(\omega) = 2\epsilon^{(n)}\omega^2\tilde{w}_0(\omega) + [f^{(n)'''}] - i\omega[f^{(n)''}] - (\omega^2 - 2)[f^{(n)'}] + i\omega(\omega^2 - 2)[f^{(n)}], \qquad n = 1, 2, 3, \cdots. \tag{19}$$

Boundedness of $\tilde{f}^{(n)}(\omega)$ requires that

$$H^{(n)}(\pm 1) = H^{(n)'}(\pm 1) = 0. \tag{20}$$

Note that the analyticity of $\tilde{w}_0(\omega)$ at $\omega = \pm 1$ is assured by condition (2).

From (16) and the continuity conditions (12)–(15) for $n = 1$ we have

$$\begin{aligned} [f^{(1)}] &= -[a^{(1)}] - [\bar{a}^{(1)}], \\ [f^{(1)'}] &= -i[a^{(1)}] + i[\bar{a}^{(1)}], \\ [f^{(1)''}] &= [a^{(1)}] + [\bar{a}^{(1)}], \\ [f^{(1)'''}] &= i[a^{(1)}] - i[\bar{a}^{(1)}], \end{aligned} \tag{21}$$

where

$$[a^{(1)}] \equiv [a^{(1)}(0, \tau)] = a^{(1)}(0^+, \tau) - a^{(1)}(0^-, \tau). \tag{22}$$

Since in general $\tilde{w}_0(1) \neq 0$, from (21) and (20) we obtain $\epsilon^{(1)} = 0$ and $[a^{(1)}(0, \tau)] = 0$. Consequently $f^{(1)}(x) \equiv 0$ and

$$u^{(1)}(x, \zeta, \tau) = a^{(1)}(\zeta, \tau) \exp(ix) + \bar{a}^{(1)}(\zeta, \tau) \exp(-ix) \tag{23}$$

with

$$[a^{(1)}(0, \tau)] = [\bar{a}^{(1)}(0, \tau)] = 0. \tag{24}$$

For $\tilde{w}_0(1) = 0$ the analysis must be modified in a manner not discussed here.

Eq. (9) for $u^{(2)}$ becomes

$$Lu^{(2)} = -2\epsilon^{(2)}w_{0xx}. \tag{25}$$

As for $u^{(1)}$, the solution may be written in the form

$$u^{(2)}(x, \zeta, \tau) = a^{(2)}(\zeta, \tau) \exp(ix) + \bar{a}^{(2)}(\zeta, \tau) \exp(-ix) + f^{(2)}(x) \tag{26}$$

where $f^{(2)}(x)$ is a particular solution of (17) with $n = 2$. The transform $\tilde{f}^{(2)}(\omega)$ satisfies Eqs. (18) and (19) for $n = 2$. The continuity requirements (12)–(15) give

$$\begin{aligned} [f^{(2)}] &= -[a^{(2)}] - [\bar{a}^{(2)}], \\ [f^{(2)'}] &= -i[a^{(2)}] + i[\bar{a}^{(2)}] - [a_\zeta^{(1)}] - [\bar{a}_\zeta^{(1)}], \\ [f^{(2)''}] &= [a^{(2)}] + [\bar{a}^{(2)}] - 2i[a_\zeta^{(1)}] + 2i[\bar{a}_\zeta^{(1)}], \\ [f^{(2)'''}] &= i[a^{(2)}] - i[\bar{a}^{(2)}] + 3[a_\zeta^{(1)}] + 3[\bar{a}_\zeta^{(1)}]. \end{aligned} \tag{27}$$

Substituting (27) into (19) and (20) for $n = 2$ gives

$$[a_\zeta^{(1)}] \equiv [a_\zeta^{(1)}(0, \tau)] = -\epsilon^{(2)}\bar{w}_0(-1)/2, \qquad [a^{(2)}] \equiv [a^{(2)}(0, \tau)] = i\epsilon^{(2)}(\bar{w}_0(-1) - \bar{w}_0'(-1))/2. \tag{28}$$

We now examine Eq. (10) for $u^{(3)}$. In order for $u^{(3)}$ to be bounded in x, quantities on the right-hand side that give rise to secular terms in x must be eliminated. Thus

$$a_{\tau\tau}^{(1)} - 4a_{\zeta\zeta}^{(1)} + a^{(1)} - 3\alpha a^{(1)}|a^{(1)}|^2 = 0. \tag{29}$$

(The complex conjugate of this equation is also asserted.) The corresponding initial conditions (derivable from (4)) and jump conditions previously stated are

$$a^{(1)}(\zeta, 0) = a_\tau^{(1)}(\zeta, 0) = 0, \tag{30}$$

$$[a^{(1)}(0, \tau)] = 0, \tag{31}$$

$$[a_\zeta^{(1)}(0, \tau)] = -\epsilon^{(2)}\bar{w}_0(-1)/2. \tag{32}$$

Now, writing the complex constant $-\epsilon^{(2)}\bar{w}_0(-1)/2$ in its polar form, namely

$$-\epsilon^{(2)}\bar{w}_0(-1)/2 = A \exp(-i\theta) \tag{33}$$

where A and θ are real, we observe that the solution to the problem (29)–(32) may be expressed as

$$a^{(1)}(\zeta, \tau) = a(\zeta, \tau) \exp(-i\theta) \tag{34}$$

where $a(\zeta, \tau)$ is real. The problem for $a(\zeta, \tau)$ consists of

$$a_{\tau\tau} - 4a_{\zeta\zeta} + a - 3\alpha a^3 = 0, \qquad \tau > 0, -\infty < \zeta < \infty, \tag{35}$$

$$a(\zeta, 0) = a_\tau(\zeta, 0) = 0, \tag{36}$$

$$[a(0, \tau)] = 0, \tag{37}$$

$$[a_\zeta(0, \tau)] = A. \tag{38}$$

By multiplying (35) by a_ζ and a_τ, simplifying, and setting $u_1 = a$, $u_2 = a_\zeta$, $u_3 = a_\tau$, we may rewrite the equation for a in the form of a quasilinear system

$$M(u) = p_\tau(\zeta, \tau, u) + q_\zeta(\zeta, \tau, u) + n(\zeta, \tau, u) = 0 \tag{39}$$

where

$$p = \begin{bmatrix} u_1 \\ 0 \\ u_3^2 + 4u_2^2 + u_1^2 - \frac{3}{2}\alpha u_1^4 \\ -2u_2u_3 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ u_1 \\ -8u_2u_3 \\ u_3^2 + 4u_2^2 - u_1^2 + \frac{3}{2}\alpha u_1^4 \end{bmatrix}, \quad n = \begin{bmatrix} -u_3 \\ -u_2 \\ 0 \\ 0 \end{bmatrix},$$

$$u = (u_1, u_2, u_3).$$

We consider the domain

$$G = \{(\zeta, \tau) \,|\, |\zeta| < 2\tau, \tau > 0\}$$

and let χ be an arbitrarily smooth test function in $R \subset G$ such that χ vanishes identically outside R. Multiplying (39) by χ and integrating over R leads to

$$\iint_R \chi M(u)\, d\zeta\, d\tau = 0. \tag{40}$$

Divide R into two regions R_1 and R_2 which are separated by the curve of discontinuity C ($\zeta = 0$):

$$R_1 = \{(\zeta, \tau) \mid 0 < \zeta < 2\tau, \tau > 0\} \cap R,$$
$$R_2 = \{(\zeta, \tau) \mid -2\tau < \zeta < 0, \tau > 0\} \cap R.$$

Thus, (40) becomes

$$\iint_{R_1} \chi M(u)\, d\zeta\, d\tau + \iint_{R_2} \chi M(u)\, d\zeta\, d\tau = 0. \tag{41}$$

We now apply Gauss' theorem separately to each of the integrals in (41) since u is smooth in R_1 and R_2. Consequently

$$\iint_{R_1} (-p\chi_\tau - q\chi_\zeta + n\chi)\, d\zeta\, d\tau + \int_{\partial R_1} \chi(pn_\tau + qn_\zeta)\, ds$$
$$+ \iint_{R_2} (-p\chi_\tau - q\chi_\zeta + n\chi)\, d\zeta\, d\tau + \int_{\partial R_2} \chi(pn_\tau + qn_\zeta)\, ds = 0, \tag{42}$$

where ∂R_1 and ∂R_2 are the boundaries of R_1 and R_2 respectively and n_τ and n_ζ represent the τ and ζ components of the outer unit normal to the boundaries of the relevant domain. We assume the existence of a generalized solution u (see [7], for example), i.e. u satisfying the equation formed by setting the sum of the first and third integrals in (42) equal to zero. Thus (42) becomes

$$\int_{\partial R_1} \chi(pn_\tau + qn_\zeta)\, ds + \int_{\partial R_2} \chi(pn_\tau + qn_\zeta)\, ds = 0. \tag{43}$$

Since χ vanishes identically outside R and since u_1, u_2, and u_3 must be bounded, then

$$\int_C \chi(\phi_\tau[p] + \phi_\zeta[q])\, ds = 0 \tag{44}$$

where C is the curve of discontinuity, ϕ_τ and ϕ_ζ are the direction cosines of the normal to C, and $[p]$, $[q]$ are the jumps in the values of p and q across C. Since χ is arbitrary, $\phi_\tau[p] + \phi_\zeta[q] = 0$. On C, $\phi_\tau = 0$ and consequently $[q] = 0$ across C. Hence

$$[u_1] = 0, \qquad [u_2 u_3] = 0,$$
$$[u_3^2 + 4u_2^2 - u_1^2 + \tfrac{3}{2}\alpha u_1^4] = 0.$$

Using the jump conditions (37) and (38) and the definition of u_j, $j = 1, 2, 3$, gives

$$a_\zeta(0^+, \tau) = -a_\zeta(0^-, \tau) = A/2, \tag{45}$$
$$a_\tau(0^+, \tau) = a_\tau(0^-, \tau) = 0. \tag{46}$$

The problem (35)–(38) for $a(\zeta, \tau)$ is now rewritten as the quarter-plane problem:

$$\begin{aligned} a_{\tau\tau} - 4a_{\zeta\zeta} + a - 3\alpha a^3 = 0, \qquad \tau > 0, \qquad \zeta > 0,& \\ a(\zeta, 0) = a_\tau(\zeta, 0) = 0,& \\ a(0^+, \tau) = \kappa, \qquad a_\zeta(0^+, \tau) = A/2,& \end{aligned} \tag{47}$$

where κ is a constant. A generalized solution to (47) can be written in the form

$$a(\zeta, \tau) = B(\zeta)H(\tau - \tfrac{1}{2}\zeta), \qquad \zeta \neq 2\tau,$$

where

$$B'' - \tfrac{1}{4}B + \tfrac{3}{4}\alpha B^3 = 0, \tag{48}$$

$$B(0^+) = \kappa, \qquad B'(0^+) = A/2, \tag{49}$$

$$H(x) = 1 \quad \text{for} \quad x > 0,$$
$$\quad = 0 \quad \text{for} \quad x < 0.$$

The solution to (35)–(38) then becomes

$$a(\zeta, \tau) = B(|\zeta|)H(\tau - \tfrac{1}{2}|\zeta|), \qquad \zeta \neq 2\tau. \tag{50}$$

We multiply (48) by B' and integrate. Applying the condition (4) which implies $B(\infty) = B'(\infty) = 0$ gives

$$(B')^2 - \tfrac{1}{4}B^2 + \tfrac{3}{8}\alpha B^4 = 0. \tag{51}$$

For $\alpha = -1$, Eq. (51) has no bounded solution. We now restrict the analysis to $\alpha = 1$ for which the structure is imperfection-sensitive.

Evaluating (51) at $\zeta = 0^+$ and using (33) and (47) gives

$$(\tfrac{1}{2}\epsilon^{(2)}|\bar{w}_0^{(1)}|)^2 = (a(0, \tau))^2 - \tfrac{3}{8}(a(0, \tau))^4.$$

We introduce the displacement measure $\sigma \equiv \delta a(0, \tau)$ (note that by (46) $a(0, \tau)$ is independent of τ); then

$$(\epsilon^{(2)})^2 = \frac{4}{|\bar{w}_0(1)|^2}\left(\frac{\sigma^2}{\delta^2} - \frac{3\sigma^4}{2\delta^4}\right).$$

Hence, keeping terms of order δ^2 in (7), we obtain

$$(1 - \lambda)\sigma^2 - \tfrac{3}{4}\sigma^4 = \tfrac{1}{8}\lambda^2\epsilon^2|\bar{w}_0^{(1)}|^2. \tag{52}$$

Eq. (5) has been used to eliminate δ. The dynamic buckling load λ_D is found by maximizing λ with respect to σ. Thus

$$\lambda_D = (1 + \sqrt{3/8}\,\epsilon\,|\bar{w}_0(1)|)^{-1} \tag{53}$$

and the critical value σ_D of σ is $\sigma_D = (\tfrac{2}{3}(1 - \lambda_D))^{1/2}$. The corresponding solution of (48) is

$$B(\zeta) = \frac{2\sqrt{3}\,(2 + \sqrt{2})\exp(-\zeta/2)}{3[1 + (3 + 2\sqrt{2})\exp(-\zeta)]}. \tag{54}$$

Hence, from (6), (23), (34), and (50) the dominant term for the deflection as $\epsilon \to 0$, $\lambda \to 1^-$ is

$$w(x, t) = \frac{4(2 + \sqrt{2})(6(1 - \lambda_D))^{1/2}H((2(1 - \lambda_D))^{1/2}(t - \tfrac{1}{2}|x|))\exp(-|x|((1 - \lambda_D)/2))^{1/2}}{3[1 + (3 + 2\sqrt{2})\exp(-|x|(2(1 - \lambda_D)))^{1/2}]} \cdot \cos(x - \theta), \tag{55}$$

where $\theta = \arg \bar{w}_0(1)$.

Concluding remarks. We recapitulate the key results. The dynamic buckling load λ_D is given by

$$\lambda_D \approx \frac{1}{1 + \sqrt{\frac{3}{8}}\, \epsilon\, |\bar{w}_0(1)|} \tag{56}$$

where

$$\bar{w}_0(1) = \int_{-\infty}^{\infty} w_0(x) \exp(ix)\, dx, \quad \text{and} \quad |w_0(x)| < M \exp(-\beta\, |x|), \qquad (M, \beta > 0).$$

The corresponding deflection is given by (55). A comparison of (56) with the static buckling load λ_s found in [6] reveals that

$$\lambda_D = \lambda_s\, . \tag{57}$$

This is a rather surprising result since previously reported analyses for modal imperfections [4] and for a *finite* column with *arbitrary* imperfections [5] produced the result $((1 - \lambda_D)/(1 - \lambda_s))^{3/2} = \sqrt{2}\, \lambda_D/\lambda_s$. However, in these analyses the condition imposed on the deflection, w say, was boundedness rather than $w \to 0$ as $|x| \to \infty$. Also, the conservative value for λ_D may be a result of the generalized nature of the solution.

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