A NONSIMILAR MOVING-WALL BOUNDARY-LAYER PROBLEM*

By JAMES E. DANBERG (*University of Delaware, Newark*)

AND KEVIN S. FANSLER (*U. S. Army Ballistic Research Laboratories, Aberdeen Proving Ground*)

1. Introduction. Rott [1] has described a nonsimilar solution for the boundary layer on a moving wall in which the wall velocity is constant and the outer edge velocity is proportional to x. Such a situation corresponds to the flow at a stagnation point on a moving wall and was derived as a limiting case of a more general problem of an oscillating wall. A counterpart to this problem may be developed where the edge velocity is constant and the wall velocity is proportional to x. It is the objective of this note to describe the latter solution and to compare its properties with that of Rott's solution.

2. Derivation. Both the steady-state stagnation point problem and its counterpart may be obtained by seeking a solution to the two-dimensional, steady, incompressible boundary layer equations in the following form:

$$u = u_e(x)f'(\eta) + u_w(x)g'(\eta), \tag{1}$$

where

$$\eta = y/h(x). \tag{2}$$

The coordinate along the surface is denoted by x and the coordinate normal to the surface by y. The boundary layer edge velocity is $u_e(x)$ and the wall velocity is $u_w(x)$. The boundary conditions are

$$f(0) = f'(0) = 0, f'(\eta) \to 1 \text{ as } \eta \to \infty$$

$$g(0) = 0, g'(0) = 1, g'(\eta) \to 0 \text{ as } \eta \to \infty.$$

The resulting equation contains nine functions of η terms and each term has a coefficient composed of various combinations of u_e , u_w , h and their derivatives; these coefficients are, therefore, only dependent on x. Thus the momentum equation may be written:

$$\sum_{i=1}^{9} X_i(x)\Phi_i(\eta) = 0.$$

where the terms in the equation are given in Table I.

* Received January 11, 1975.

TABLE I. Terms in the equation of motion.

Term	X_i Coefficient	$\Phi_i(\eta)$
1	$\nu u_e/h^2$	f'''
2	u_e^2h'/h	ff''
3	$u_e u_e'$	$ff'' - f'^2 + 1$
4	$u_w u_e'$	$fg'' - f'g'$
5	$u_w u_e h'/h$	$f''g + g''f$
6	$u_e u_w'$	$gf'' - f'g'$
7	$u_w u_w'$	$gg'' - g'^2$
8	u_w^2h'/h	gg''
9	$\nu u_w/h^2$	g'''

Note: Primes in X_i terms indicate differentiation with respect to x and primes in Φ_i terms indicate differentiation with respect to η.

u_w = *Constant.* Consider first the situation where u_w is constant. The coefficients of terms 6 and 7 are identically zero and thus without loss of generality, terms 1, 2 and 3 may be combined to form:

$$(\nu u_e/h^2)\{f''' - A_1[ff'' + \beta_1(1 - f'^2)]\}.$$

The terms within the outer brackets may be recognized as the left-hand side of the Falkner-Skan equation when $A_1 = -1$. If f is assumed to be satisfied by the Falkner-Skan solution then:

$$u_e = Ux^{\beta_1/(2-\beta_1)}, \ h = [(2 - \beta_1)\nu x/u_e]^{1/2}.$$

The remaining terms 4, 5, 8 and 9 have coefficients which allow the formation of an ordinary differential equation for g if h and u_e' are constants. Thus the external flow corresponds to a two-dimensional stagnation point with $\beta_1 = 1$ and $u_e = u_e'(0)x$. The equation which g must satisfy becomes

$$g''' + (fg'' - f'g') = 0, \tag{3}$$

which is identical to the equation obtained by Rott as the limiting steady case of a two-dimensional stagnation point on an oscillating wall.

The g function was shown by Rott to be related to the f function such that:

$$g' = f''/f''(0),$$

which when inserted into Eq. (3) may be shown to give the Falkner-Skan equation for $\beta_1 = 1$ differentiated once with respect to η.

The f' and g' functions for this case are shown in Fig. 1.

u_e = *Constant.* The second problem apparently has not been considered before. That is, by taking u_e = constant then terms 3 and 4 are zero and terms containing only functions of g may be combined to give

$$(u_w\nu/h^2)[g''' + A_2(gg'' - \beta_2 g'^2)].$$

If A_2 and β_2 are constants, the quantity within the outer brackets is a generalization of Sakiadis's [2] continuous surface boundary layer with $A_2 = 1$; thus

$$u_w = Ux^{\beta_2/(2-\beta_2)} \quad \text{and} \quad h = [(2 - \beta_2)\nu x/u_w]^{1/2}.$$

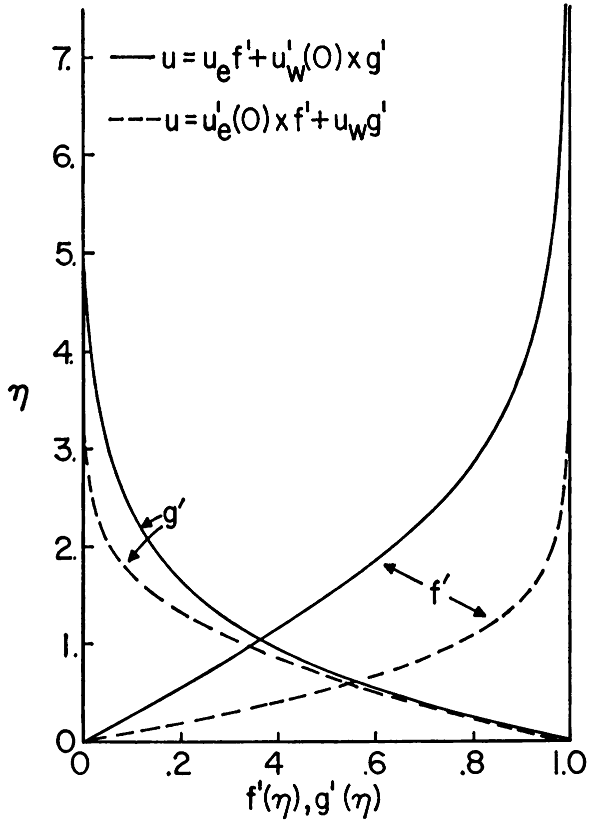

FIG. 1. Velocity profile functions f' and g' for moving wall boundary layer.

The remaining coefficients of the f and mixed f and g terms may be reduced to an ordinary differential equation if $u_w = u_w'(0)x$ and $h =$ constant and $\beta_2 = 1$. The following set of equations must be solved for the $u_e =$ constant case:

$$g''' + gg'' - g'^2 = 0, \tag{4}$$

$$f''' + f''g - g'f' = 0. \tag{5}$$

Eq. (4) has the rather simple solution:

$$g = 1 - \exp(-\eta), \tag{6}$$

which also satisfies the boundary conditions. The f function is not as simple but may be obtained as follows. It may be recognized that $f' = g'$ is a solution to Eq. (5) although it does not satisfy the boundary conditions. This solution permits the use of the method of variation of coefficients to obtain the second linearly independent solution which is, after use of the outer boundary condition,

$$\begin{aligned} f' &= \exp(-\eta) \int_0^\eta \exp[\eta_1 - \exp(-\eta_1)]\, d\eta_1. \\ f' &= \exp(-z) - zE_1(z) - z[\exp(-1) - E_1(1)]. \end{aligned} \tag{7}$$

where $z = \exp(-\eta)$ and $E_1(z) = \int_z^\infty t^{-1} \exp(-t)\, dt$. Also f can be easily obtained using the exponential-integral relationship $dE_\eta(z)/dz = -E_{\eta-1}(z)$. The f' and g' functions for the u_e = constant case are also shown in Fig. 1. Fig. 2 compares the velocity profiles for the two nonsimilar problems, u_w = constant and u_e = constant, as a function of the wall and edge velocity.

A skin friction parameter may be defined for these developing profiles,

$$c_f h(u_w - u_e)/(2\nu) \equiv \frac{h}{u_w - u_e} \left.\frac{\partial u}{\partial y}\right|_{y=0} = Bf''(0) + (B+1)g''(0).$$

The right-hand side is linear in B, a wall-to-edge-velocity ratio parameter:

$$B = u_e/(u_w - u_e).$$

Note that vanishing of the skin friction does not indicate separation for these problems as it does for the non-moving-wall situation. The displacement thickness is also linear in B and is

$$\delta_1^* \equiv -\delta^*/h = Bf_d(\infty) + (B+1)g(\infty),$$

where the following definitions are used:

$$\delta^* = \int_0^\infty [(u_e - u)/(u_w - u_e)]\, dy,$$

$$f_d(\infty) = \lim_{\eta\to\infty} (f - \eta).$$

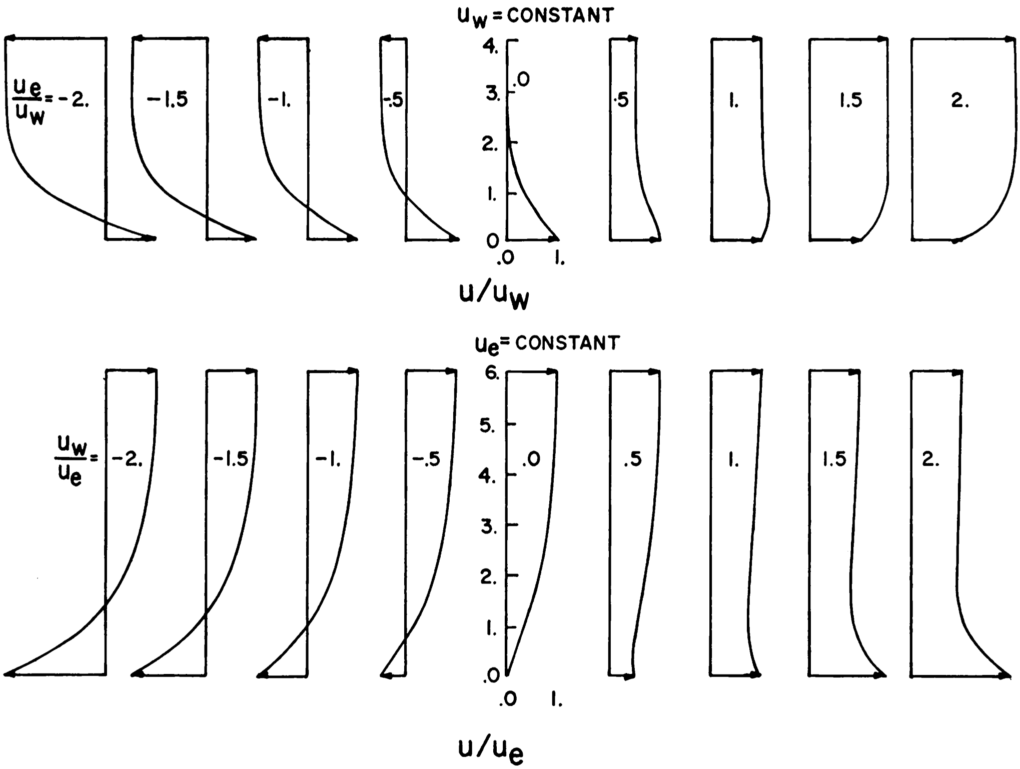

FIG. 2. Velocity profile variation for u_w = constant and u_e = constant.

The momentum thickness is quadratic in B and somewhat more complicated because of the difficulty in evaluating integrals of f'^2 and g'^2. Thus,

$$\theta_1 \equiv -\theta/h = (1 + B)^2\theta_g + B(1 + B)(2\theta_{fg} - g[\infty]) - B^2\theta_f$$

where

$$\theta = \int_0^\infty [u(u_e - u)/(u_w - u_e)^2]\, dy$$

and

$$\theta_g = \int_0^\infty g'^2\, d\eta, \tag{8}$$

$$\theta_f = \int_0^\infty f'(1 - f')\, d\eta, \tag{9}$$

$$\theta_{fg} = \int_0^\infty f'g'\, d\eta. \tag{10}$$

It should be noted that in both cases integrals (8)–(10) may be evaluated solely from the solution of the f equation. For example, in the u_w = constant case, $g(\infty) = 1/f''(0)$, $g''(0) = -1/f''(0)$ and integration of (3) with respect to η gives $\theta_{fg} = 1/2f''(0)$. For the case of u_e = constant, $\theta_{fg} = .5(1 - f''(0))$ may be obtained from (5).

The integrals required to define the variation with B of the above parameters are given in Table II.

TABLE II.

Term	u_e = Constant	u_w = Constant
$g''(0)$	−1.0000	− .8113
$f''(0)$	.3679	1.2326
$g(\infty)$	1.0000	.8113
$f_d(\infty)$	−1.796	− .6479
θ_f	.7773	.2923
θ_{fg}	.3160	.4056
θ_g	.5000	.4692

The u_e = constant solution provides a second numerical solution to a nonsimilar moving wall problem with which to test approximate methods of solving the moving wall boundary-layer equations.

REFERENCES

[1] N. Rott, *Unsteady viscous flow in the vicinity of a stagnation point,* Quart. Appl. Math. **13**, 444–451 (1956)

[2] B. C. Sakiadis, *Boundary layer behavior on continuous solid surfaces: I. Boundary layer equations for two-dimensional and axisymmetric flow,* AIChE Journal **7**, 26–28 (1961); *II. The boundary layer on a continuous flat surface,* AIChE Journal **7**, 221–225 (1961); *III. The boundary layer on a continuous cylindrical surface,* AIChE Journal **7**, 467–472 (1961)