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GURTIN-TYPE PROPERTIES ASSOCIATED WITH WAVE PROPAGATION IN A VISCOUS, HEAT-CONDUCTING GAS*

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Summary. Variational and reciprocity principles of the Gurtin type are established for the linear initial-boundary value problems associated with small-amplitude wave motion in a viscous, heat-conducting gas.

1. Introduction. The method of time convolutions due to Gurtin [1, 2, 3] has been employed by several authors in the recent literature [4, 5, 6] to derive variational and reciprocity principles associated with the classical heat and wave equations. The initial-boundary value problems investigated apply to a wide range of phenomena in continuum mechanics. In the context of the acoustic problem for a gas, however, the classical wave equation is valid only under the assumption that viscous dissipation and heat conduction are neglected. It seems from the literature that these effects have not been taken into account in the various formulations to date.

In this paper an initial-boundary value problem is formulated which governs the small-amplitude motion of a viscous, heat-conducting gas. Variational and reciprocity principles of the Gurtin type are established for this general problem. These Gurtin-type principles are simplified further by the methods developed recently by Herrera and Bielak in [6]. Moreover, it is shown that the general problem can be replaced by two initial-boundary value problems. Variational and reciprocity principles are also given for these latter problems.

2. Formulation of the general problem. A viscous heat-conducting gas occupies the open bounded domain D of the n-dimensional Euclidean space R^n. The closure and boundary of D are denoted respectively by $\bar{D}$ and ∂D, with $x = (x_1, \cdots x_n)$ denoting a point in $\bar{D}$. The undisturbed, uniform gas pressure, density and temperature are given by p_0, ρ_0, T_0 and the specific heats and thermal conductivity are given by C_v, C_p and κ.

The gas undergoes a small-amplitude motion initiated at time $t = 0$ in which the velocity, temperature, pressure and density distributions are given by $\mathbf{q} = (u_1, \cdots u_n)$, T, p and ρ respectively. The fractional changes in density and temperature are s and η, with $\rho = \rho_0(1 + s)$ and $T = T_0(1 + \eta)$. If the n-dimensional gradient and Laplacian operators are denoted by ∇ and Δ, then the governing equations of continuity, momen-

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tum, energy and state have the form (e.g. [7])

$$\nabla \cdot \mathbf{q} = -\frac{\partial s}{\partial t} + f, \qquad (x, t) \in D \times [0, \infty), \tag{2.1}$$

$$\rho_0 \frac{\partial \mathbf{q}}{\partial t} = -\nabla p + \mu \Delta \mathbf{q} + \frac{\mu}{3} \nabla \theta, \qquad (x, t) \in D \times [0, \infty), \tag{2.2}$$

$$\frac{\partial \eta}{\partial t} - (\gamma - 1) \frac{\partial s}{\partial t} = \nu' \Delta \eta + g, \qquad (x, t) \in D \times [0, \infty), \tag{2.3}$$

$$\theta = \nabla \cdot \mathbf{q}, \qquad (x, t) \in D \times [0, \infty), \tag{2.4}$$

$$p = p_0(1 + s + \eta), \qquad (x, t) \in D \times [0, \infty), \tag{2.5}$$

where μ is the viscosity coefficient, $\gamma = C_p/C_v$ and $\nu' = \kappa/\rho_0 C_v$. We also allow for the possibility of mass and heat sources by introducing the prescribed source functions $f(x, t)$ and $g(x, t)$.

The initial conditions imposed on $\mathbf{q}$, s and η are

$$\mathbf{q}(x, 0) = \mathbf{q}_0(x) \equiv (u_{1_0}, \cdots, u_{n_0}), \qquad x \in D, \tag{2.6}$$

$$s(x, 0) = s_0(x), \qquad x \in D, \tag{2.7}$$

$$\eta(x, 0) = \eta_0(x), \qquad x \in D. \tag{2.8}$$

On the boundary ∂D, which we consider as the disjoint union of ∂D_i , $i = 1, 2, 3$, we impose the following conditions:

$$\mathbf{q} = \mathbf{Q}(x, t) \equiv (U_1, \cdots, U_n), \qquad (x, t) \in \partial D \times (0, \infty), \tag{2.9}$$

$$\eta = f_1(x, t), \qquad (x, t) \in \partial D_1 \times (0, \infty), \tag{2.10}$$

$$\frac{\partial \eta}{\partial \nu} = f_2(x, t), \qquad (x, t) \in \partial D_2 \times (0, \infty), \tag{2.11}$$

$$\frac{\partial \eta}{\partial \nu} + k(x)\eta = f_3(x, t), \qquad (x, t) \in \partial D_3 \times (0, \infty), \tag{2.12}$$

where $\mathbf{q}_0$, s_0 , η_0 , $\mathbf{Q}$, f_i , $i = 1, 2, 3$ and k are prescribed, with the outward normal to ∂D denoted by $\boldsymbol{\nu}$. All functions appearing above and in what follows are real-valued functions of position x and time t defined on $\bar{D} \times [0, \infty)$ and we will assume that they are sufficiently well behaved and ∂D sufficiently smooth to justify the operations in the subsequent analysis.

Before proceeding to the derivation of variational and reciprocity principles, we require the following definitions: for two scalar functions $a(x, t)$ and $b(x, t)$ the convolution is defined in the usual way by

$$a^*b = \int_0^t a(x, t')b(x, t - t')\, dt'. \tag{2.13}$$

If $\mathbf{a} \equiv (a_1, \cdots, a_n)$, $\mathbf{b} \equiv (b_1, \cdots, b_n)$ then

$$\mathbf{a}^* \cdot \mathbf{b} = \sum_{i=1}^{n} a_i{}^* b_i . \tag{2.14}$$

The algebraic properties of the convolution are well known and need not be stated here.

3. A variational principle. To derive a variational principle which characterizes the initial-boundary value problem (2.1)–(2.12) it is convenient, following Gurtin [1], to replace the above problem by an equivalent boundary-value problem. This boundary-value problem is obtained by integrating Eqs. (2.1)–(2.3) with respect to time and using the initial conditions (2.6)–(2.8). The resulting set of integrodifferential equations is

$$s - s_0 + 1*(\nabla\cdot\mathbf{q}) - 1*f = 0, \qquad (x, t) \in D \times [0, \infty), \tag{3.1}$$

$$\rho_0(\mathbf{q} - \mathbf{q}_0) = -1*(\nabla p) + \mu 1*(\Delta\mathbf{q}) + \frac{\mu}{3} 1*(\nabla\theta), \qquad (x, t) \in D \times [0, \infty), \tag{3.2}$$

$$\eta - \eta_0 - (\gamma - 1)(s - s_0) = \nu' 1*(\Delta\eta) + 1*g, \qquad (x, t) \in D \times [0, \infty). \tag{3.3}$$

By using the algebraic properties of the convolution it is not difficult to show that Eqs. (3.1)–(3.3) are equivalent to (2.1)–(2.3) together with (2.6)–(2.8). The equivalent boundary-value problem then consists of Eqs. (3.1)–(3.3) together with (2.4), (2.5) and the boundary conditions (2.9)–(2.12).

Let $\mathbf{q}$, p, s, η, θ and their spatial derivatives belong to a function space L. For each $t \in [0, \infty)$ define the functional $U_t(\mathbf{q}, p, s, \eta, \theta)$ on L by the relation

$$\begin{aligned}
U_t(\mathbf{q}, p, s, \eta, \theta) = {} & \tfrac{1}{2}\rho_0 \int_D (\mathbf{q} - 2\mathbf{q}_0)*\cdot\mathbf{q}\, d\tau \\
& + \int_D 1*(\nabla p)*\cdot\mathbf{q}\, d\tau - \frac{\mu}{3}\int_D 1*(\nabla\theta)*\cdot\mathbf{q}\, d\tau \\
& + \frac{\mu}{2}\sum_{i=1}^{n}\int_D 1*(\nabla u_i)*\cdot(\nabla u_i)\, d\tau \\
& + \int_D (1*f*p - s*p + s_0*p)\, d\tau - \frac{\mu}{6}\int_D 1*\theta*\theta\, d\tau \\
& + p_0\int_D (1 + \tfrac{1}{2}s + \eta)*s\, d\tau \\
& - \tfrac{1}{2}p_0\nu'(\gamma - 1)^{-1}\int_D 1*(\nabla\eta)*\cdot(\nabla\eta)\, d\tau - p_0\int_D s_0*\eta\, d\tau \\
& - p_0(\gamma - 1)^{-1}\int_D (\tfrac{1}{2}\eta - \eta_0)*\eta\, d\tau + p_0(\gamma - 1)^{-1}\int_D 1*g*\eta\, d\tau \\
& - \mu\int_{\partial D} 1*(\mathbf{q} - \mathbf{Q})*\cdot\frac{\partial\mathbf{q}}{\partial\nu}\, d\sigma - \int_{\partial D} 1*(\boldsymbol{\nu}\cdot\mathbf{Q})*p\, d\sigma \\
& + \frac{\mu}{3}\int_{\partial D} 1*(\boldsymbol{\nu}\cdot\mathbf{Q})*\theta\, d\sigma + p_0\nu'(\gamma - 1)^{-1}\int_{\partial D_1} 1*(\eta - f_1)*\frac{\partial\eta}{\partial\nu}\, d\sigma \\
& + p_0\nu'(\gamma - 1)^{-1}\int_{\partial D_2} 1*f_2*\eta\, d\sigma \\
& + p_0\nu'(\gamma - 1)^{-1}\int_{\partial D_3} 1*(f_3 - \tfrac{1}{2}k\eta)*\eta\, d\sigma, \qquad t \in [0, \infty);
\end{aligned} \tag{3.4}$$

then

$$\delta U_t(\mathbf{q}, p, s, \eta, \theta) = 0 \text{ on } L, \qquad t \in [0, \infty), \tag{3.5}$$

iff $(\mathbf{q}, p, s, \eta, \theta)$ is a solution of the initial-boundary value problem (2.1)–(2.12).

Proof: Adopting the customary procedure of the variational calculus we find, after using the algebraic properties of the convolution and the divergence theorem,

$$\begin{aligned}
\delta U_t(\mathbf{q}, p, s, \eta, \theta) = &\int_D \left(\rho_0(\mathbf{q} - \mathbf{q}_0) + 1*(\nabla p) - \mu 1*(\Delta \mathbf{q}) - \frac{\mu}{3} 1*(\nabla \theta)\right)^* \cdot \delta \mathbf{q} \, d\tau \\
&+ \int_D (1*f + s_0 - s - 1*(\nabla \cdot \mathbf{q}))*\delta p \, d\tau + \frac{\mu}{3} \int_D 1*(\nabla \cdot \mathbf{q} - \theta)*\delta\theta \, d\tau \\
&+ \int_D (p_0(1 + s + \eta) - p)*\delta s \, d\tau \\
&+ p_0(\gamma - 1)^{-1} \int_D ((\gamma - 1)(s - s_0) + \nu' 1*(\Delta \eta) \\
&- \eta + \eta_0 + 1*g)*\delta\eta \, d\tau - \mu \int_{\partial D} 1*(\mathbf{q} - \mathbf{Q})*\cdot\left(\frac{\partial \delta \mathbf{q}}{\partial \nu}\right) d\sigma \\
&+ \int_{\partial D} 1*(\mathbf{v}\cdot(\mathbf{q} - \mathbf{Q}))*\delta p \, d\sigma - \frac{\mu}{3} \int_{\partial D} 1*(\mathbf{v}\cdot(\mathbf{q} - \mathbf{Q}))*\delta\theta \, d\sigma \\
&+ p_0\nu'(\gamma - 1)^{-1} \int_{\partial D_1} 1*(\eta - f_1)*\left(\frac{\partial}{\partial \nu} \delta\eta\right) d\sigma \\
&+ p_0\nu'(\gamma - 1)^{-1} \int_{\partial D_2} 1*\left(f_2 - \frac{\partial \eta}{\partial \nu}\right)^* \delta\eta \, d\sigma \\
&+ p_0\nu'(\gamma - 1)^{-1} \int_{\partial D_3} 1*\left(f_3 - k\eta - \frac{\partial \eta}{\partial \nu}\right)^* \delta\eta \, d\sigma, \qquad t \in [0, \infty). \qquad (3.6)
\end{aligned}$$

If $(\mathbf{q}, p, s, \eta, \theta)$ is a solution of the initial-boundary value problem (2.1)–(2.12) then the equivalent boundary value problem is satisfied and (3.5) holds. Conversely, if (3.5) holds then, setting

$$\delta p, \delta\theta, \delta s, \delta\eta, \delta u_2 \cdots \delta u_n = 0 \text{ in } D \times [0, \infty);$$

$$\frac{\partial}{\partial \nu} \delta \mathbf{q}, \delta p, \delta\theta = 0 \quad \text{on} \quad \partial D \times [0, \infty); \qquad \frac{\partial}{\partial \nu} \delta\eta = 0 \quad \text{on} \quad \partial D_1 \times [0, \infty);$$

$\delta\eta = 0$ on $(\partial D_2 \cup \partial D_3) \times [0, \infty)$ and $\delta u_1 \not\equiv 0$ in $D \times [0, \infty)$, we obtain

$$\int_D \left(\rho_0(u_1 - u_{10}) + 1*\left(\frac{\partial p}{\partial x_1}\right) - u 1*(\Delta \mu_1) - \frac{\mu}{3} 1*\left(\frac{\partial \theta}{\partial x_1}\right)\right)^* \delta u_1 \, d\tau = 0, \quad t \in [0, \infty). \quad (3.7)$$

By the fundamental lemma of the variational calculus [1] the first of the n equations (3.2) is satisfied. Again, by setting $\delta p, \delta\theta, \delta s, \delta\eta, \delta u_3 \cdots \delta u_n = 0$ in

$$D \times [0, \infty); \qquad \frac{\partial}{\partial \nu} \delta \mathbf{q}, \delta p, \delta\theta = 0$$

on

$$\partial D \times [0, \infty); \qquad \frac{\partial}{\partial \nu} \delta\eta = 0 \quad \text{on} \quad \partial D_1 \times [0, \infty); \qquad \delta\eta = 0$$

on $(\partial D_2 \cup \partial D_3) \times [0, \infty)$ and $\delta u_2 \not\equiv 0$ in $D \times [0, \infty)$ and using the fundamental lemma of the variational calculus, we will obtain the second of the n equations (3.2). Continuing in this way, all the equations (3.2) can be obtained. Similarly, with the choice $\delta\theta$, δs, $\delta\eta = 0$ in

$$D \times [0, \infty); \qquad \frac{\partial}{\partial\nu}\,\delta\mathbf{q},\ \delta p,\ \delta\theta = 0 \quad \text{on} \quad \partial D \times [0, \infty);$$

$$\frac{\partial}{\partial\nu}\,\delta\eta = 0 \quad \text{on} \quad \partial D_1 \times [0, \infty); \qquad \delta\eta = 0$$

on $(\partial D_2 \cup \partial D_3) \times [0, \infty)$ and $\delta p \not\equiv 0$ in $D \times [0, \infty)$ we obtain Eq. (3.1). The remaining field equations (2.4), (2.5) and (3.3) can be obtained by making the appropriate choice of

$$\delta\theta,\ \delta s,\ \delta\eta \quad \text{in} \quad \bar{D} \times [0, \infty) \quad \text{and} \quad \frac{\partial}{\partial\nu}\,\delta\mathbf{q},\ \delta p,\ \frac{\partial}{\partial\nu}\,\delta\eta,\ \delta\eta \quad \text{on} \quad \partial D \times [0, \infty)$$

and again appealing to the fundamental lemma of the variational calculus. Also, by carefully choosing

$$\frac{\partial}{\partial\nu}\,\delta\mathbf{q},\ \delta p,\ \delta\theta,\ \frac{\partial}{\partial\nu}\,\delta\eta \quad \text{and} \quad \delta\eta \quad \text{on} \quad \partial D \times [0, \infty)$$

we can, in the same manner as that adopted for the field equations, obtain all the boundary conditions (2.9)–(2.12). All the equations of the boundary-value problem (3.1)–(3.3), (2.4), (2.5) and (2.9)–(2.12) are now established so that Eqs. (2.11)–(2.12) hold and the proof is complete.

It has been proved above that a variational principle of the Gurtin type exists which characterizes the general initial-boundary value problem (2.1)–(2.12) and for which the initial and boundary conditions are natural. However, it has been shown recently by Herrera and Bielak [6] that variational principles of the Gurtin type can be simplified by reducing the number of convolutions used in the construction of the functionals. In the present investigation we can accomplish this by introducing the functional $\bar{U}_t$ where

$$1 * \bar{U}_t \equiv U_t \tag{3.8}$$

Since $\delta U_t = 0$ iff $\delta\bar{U}_t = 0$, the variational principle (3.5) can be restated in the same form with U_t replaced by $\bar{U}_t$. The functional $\bar{U}_t$ can be found by differentiating (3.8) with respect to time using (3.3). We obtain

$$\begin{aligned}
\bar{U}_t \equiv \frac{d}{dt} U_t = {} & \tfrac{1}{2}\rho_0 \int_D (\mathbf{q}(x, 0)\cdot\mathbf{q}(x, t) + \mathbf{q}^*\cdot\dot{\mathbf{q}} - 2\mathbf{q}_0\cdot\mathbf{q}(x, t))\, d\tau \\
& + \int_D (\nabla p)^*\cdot\mathbf{q}\, d\tau - \frac{\mu}{3}\int_D (\nabla\theta)^*\cdot\mathbf{q}\, d\tau + \frac{\mu}{2}\sum_{i=1}^{n}\int_D (\nabla u_i)^*\cdot(\nabla u_i)\, d\tau \\
& + \int_D (f^* p - s^*\dot{p} - p(x, 0)s(x, t) + s_0 p(x, t))\, d\tau - \frac{\mu}{6}\int_D \theta^*\theta\, d\tau \\
& + p_0 \int_D (s + \tfrac{1}{2}(s^*\dot{s} + s(x, 0)s(x, t) + \eta^*\dot{s} + s(x, 0)\eta(x, t))\, d\tau \\
& - \tfrac{1}{2}p_0\nu'(\gamma - 1)^{-1}\int_D (\nabla\eta)^*\cdot(\nabla\eta)\, d\tau - p_0\int_D s_0\eta(x, t)\, d\tau
\end{aligned}$$

$$- p_0(\gamma - 1)^{-1} \int_D (\tfrac{1}{2}(\eta^* \dot{\eta} + \eta(x, 0)\eta(x, t) - \eta_0 \eta(x, t)) \, d\tau$$

$$+ p_0(\gamma - 1)^{-1} \int_D g^* \eta \, d\tau - \mu \int_{\partial D} (\mathbf{q} - \mathbf{Q})^* \cdot \frac{\partial \mathbf{q}}{\partial \nu} \, d\sigma$$

$$- \int_{\partial D} (\mathbf{v} \cdot \mathbf{Q})^* p \, d\sigma + \frac{\mu}{3} \int_{\partial D} (\mathbf{v} \cdot \mathbf{Q})^* \theta \, d\sigma + p_0 \nu'(\gamma - 1)^{-1} \int_{\partial D_1} (\eta - f_1)^* \frac{\partial \eta}{\partial \nu} \, d\sigma$$

$$+ p_0 \nu'(\gamma - 1)^{-1} \int_{\partial D_2} f_2^* \eta \, d\sigma + p_0 \nu'(\gamma - 1) \quad \int_{\partial D_3} (f_3 - \tfrac{1}{2} k\eta)^* \eta \, d\sigma, \; t \in [0, \infty), \tag{3.9}$$

where the dot notation is used to denote partial differentiation with respect to time. In concluding this section we note that the variational principle associated with the functional $\bar{U}_t$ above can also be proved by proceeding in a similar manner as that adopted for the variational principle associated with U_t.

4. The reciprocity principle. Associated with the variational principle derived above there is a reciprocity principle which can be derived in the following manner. We write $\mathbf{q} = \mathbf{q}' + \mathbf{q}''$, $p = p' + p''$, $s = s' + s''$, $\eta = \eta' + \eta''$, $\theta = \theta' + \theta''$ where $\mathbf{q}''$, p'', s'', η'' and θ'' satisfy the following equations:

$$\nabla \cdot \mathbf{q}'' = - \frac{\partial s''}{\partial t}, \qquad (x, t) \in D \times [0, \infty), \tag{4.1}$$

$$\rho_0 \frac{\partial \mathbf{q}''}{\partial t} = - \nabla p'' + \mu \Delta \mathbf{q}'' + \frac{\mu}{3} \nabla \theta'', \qquad (x, t) \in D \times [0, \infty), \tag{4.2}$$

$$\frac{\partial \eta''}{\partial t} - (\gamma - 1) \frac{\partial s''}{\partial t} = \nu' \Delta \eta'', \qquad (x, t) \in D \times [0, \infty), \tag{4.3}$$

$$\theta'' = \nabla \cdot \mathbf{q}'', \qquad (x, t) \in D \times [0, \infty), \tag{4.4}$$

$$p'' = p_0(1 + s'' + \eta''), \qquad (x, t) \in D \times [0, \infty), \tag{4.5}$$

together with the initial and boundary conditions (2.6)–(2.12). The functions $\mathbf{q}'$, p', s', η' and θ' are then determined by the following initial-boundary value problem:

$$\nabla \cdot \mathbf{q}' = - \frac{\partial s'}{\partial t} + f, \qquad (x, t) \in D \times [0, \infty), \tag{4.6}$$

$$p_0 \frac{\partial \mathbf{q}'}{\partial t} = - \nabla p' + \mu \Delta \mathbf{q}' + \frac{\mu}{3} \nabla \theta', \qquad (x, t) \in D \times [0, \infty), \tag{4.7}$$

$$\frac{\partial \eta'}{\partial t} - (\gamma - 1) \frac{\partial s'}{\partial t} = \gamma' \Delta \eta' + g, \qquad (x, t) \in D \times [0, \infty), \tag{4.8}$$

$$\theta' = \nabla \cdot \mathbf{q}', \qquad (x, t) \in D \times [0, \infty), \tag{4.9}$$

$$p' = p_0(s' + \eta'), \qquad (x, t) \in D \times [0, \infty), \tag{4.10}$$

$$\mathbf{q}'(x, 0) = 0, \qquad x \in D, \tag{4.11}$$

$$s'(x, 0) = 0, \qquad x \in D, \tag{4.12}$$

$$\eta'(x, 0) = 0, \qquad x \in D, \tag{4.13}$$

$$\mathbf{q}' = 0, \qquad (x, t) \in \partial D \times (0, \infty), \tag{4.14}$$

$$\eta' = 0, \qquad (x, t) \in \partial D_1 \times (0, \infty), \tag{4.15}$$

$$\partial\eta'/\partial\nu = 0, \qquad (x, t) \in \partial D_2 \times (0, \infty), \tag{4.16}$$

$$(\partial\eta'/\partial\nu) + k(x)\eta' = 0, \qquad (x, t) \in \partial D_3 \times (0, \infty). \tag{4.17}$$

If $(\mathbf{q}_1, p_1, s_1, \eta_1, \theta_1)$ and $(\mathbf{q}_2, p_2, s_2, \eta_2, \theta_2)$ are solutions of the initial-boundary value problem (2.1)–(2.12) corresponding to the pairs of source functions $(f^{(1)}, g^{(1)})$ and $(f^{(2)}, g^{(2)})$ respectively, then there exists a reciprocity principle of the form

$$\int_D \{(s_1' + \eta_1')*f^{(2)} + (\gamma - 1)^{-1}\eta_1'*g^{(2)}\}\, d\tau$$

$$= \int_D \{(s_2' + \eta_2')*f^{(1)} + (\gamma - 1)^{-1}\eta_2'*g^{(1)}\}\, d\tau, \qquad t \in [0, \infty). \tag{4.18}$$

Proof: Since the boundary and initial conditions are the same for both

$$(\mathbf{q}_1, p_1, s_1, \eta_1, \theta_1) \quad \text{and} \quad (\mathbf{q}_2, p_2, s_2, \eta_2, \theta_2),$$

we need only consider the initial-boundary value problem (4.6)–(4.17) to establish the reciprocity principle (4.18).

The equations equivalent to (4.6)–(4.8) and (4.11)–(4.13) can be written in the form

$$1*(\nabla\cdot\mathbf{q}') = -s + 1*f, \qquad (x, t) \in D \times [0, \infty), \tag{4.19}$$

$$\rho_0\mathbf{q}' = -1*(\nabla p') + \mu 1*(\Delta\mathbf{q}') + \frac{\mu}{3}1*(\nabla\theta'), \qquad (x, t) \in D \times [0, \infty), \tag{4.20}$$

$$\eta' - (\gamma - 1)s' = \nu' 1*(\Delta\eta') + 1*g, \qquad (x, t) \in D \times [0, \infty). \tag{4.21}$$

If $(\mathbf{q}_i', p_i', s_i', \eta_i', \theta_i')$, $i = 1, 2$ are solutions of (4.6)–(4.17) corresponding to the pairs of source functions $(f^{(i)}, g^{(i)})$, $i = 1, 2$, then from (4.20) we obtain

$$0 = \int_D \rho_0(\mathbf{q}_1'*\cdot\mathbf{q}_2' - \mathbf{q}_2'*\cdot\mathbf{q}_1')\, d\tau = \int_D \{-1*((\nabla p_1')*\cdot\mathbf{q}_2' - (\nabla p_2')*\cdot\mathbf{q}_1')$$

$$+ \mu 1*((\Delta\mathbf{q}_1')*\cdot\mathbf{q}_2 - (\Delta\mathbf{q}_2')*\cdot\mathbf{q}_1') + \frac{\mu}{3}1*((\nabla\theta_1')*\cdot\mathbf{q}_2' - (\nabla\theta_2')*\cdot\mathbf{q}_1')\}\, d\tau. \tag{4.22}$$

Using the divergence theorem, Eq. (4.9) and the boundary condition (4.14), we find that (4.22) can be written in the form

$$0 = \int_D 1*\{p_1'*(\nabla\cdot\mathbf{q}_2') - p_2'*(\nabla\cdot\mathbf{q}_1')\}\, d\tau. \tag{4.23}$$

Differentiating (4.23) with respect to time and using (4.9)–(4.10), we obtain the relation

$$\int_D (s_1' + \eta_1')*\theta_2'\, d\tau = \int_D (s_2' + \eta_2')*\theta_1'\, d\tau. \tag{4.24}$$

With the aid of (4.9), (4.19) and (4.24) we can write

$$0 = \int_D 1*\{\theta_1'*(s_2' + \eta_2') - \theta_2'*(s_1' + \eta_1')\}\, d\tau$$

$$= - \int_D \{s_1'*(s_2' + \eta_2') - s_2'*(s_1' + \eta_1')\}\, d\tau$$

$$+ \int_D 1*\{f^{(1)}*(s_2' + \eta_2') - f^{(2)}*(s_1' + \eta_1')\}\, d\tau, \qquad (4.25)$$

so that

$$\int_D (s_1'*\eta_2' - s_2'*\eta_1')\, d\tau = \int_D 1*\{f^{(1)}*(s_2' + \eta_2') - f^{(2)}*(s_1' + \eta_1')\}\, d\tau. \qquad (4.26)$$

From Eq. (4.21) we obtain

$$(\gamma - 1) \int_D (s_1'*\eta_2' - s_2'*\eta_1')\, d\tau = \nu' \int_D 1*(\eta_1'*(\Delta\eta'_2) - \eta_2'*(\Delta\eta'_1))\, d\tau$$

$$+ \int_D 1*(\eta_1'*g^{(2)} - \eta_2'*g^{(1)})\, d\tau. \qquad (4.27)$$

Again, using the divergence theorem together with the boundary conditions (4.15)–(4.17), we find

$$(\gamma - 1) \int_D (s_1'*\eta_2' - s_2'*\eta_1')\, d\tau = \int_D 1*(\eta_1'*g^{(2)} - \eta_2'*g^{(1)})\, d\tau. \qquad (4.28)$$

Combining (4.26) and (4.28) we obtain, after differentiating with respect to time, the reciprocity principle

$$\int_D \{(s_1' + \eta_1')*f^{(2)} + (\gamma - 1)^{-1}\eta_1'*g^{(2)}\}\, d\tau$$

$$= \int_D \{(s_2' + \eta_2')*f^{(1)} + (\gamma - 1)^{-1}\eta_2'*g^{(1)}\}\, d\tau, \qquad t \in [0, \infty). \qquad (4.29)$$

Finally we note two special cases of (4.29). First, *in the absence of heat sources* $g^{(i)} = 0$, $i = 1, 2$ and the reciprocity principle (4.29) reduces to

$$\int_D (s_1' + \eta_1')*f^{(2)}\, d\tau = \int_D (s_2' + \eta_2')*f^{(1)}\, d\tau, \qquad t \in [0, \infty), \qquad (4.30)$$

which is similar to the reciprocity principle obtained in [4] for the classical wave equation. Eq. (4.30) implies that such a reciprocity principle also holds for a viscous, heat-conducting gas in the absence of heat sources. Again, *in the absence of mass sources* $f^{(i)} = 0$, $i = 1, 2$ and the reciprocity principle (4.29) reduces to

$$\int_D \eta_1'*g^{(2)}\, d\tau = \int_D \eta_2'*g^{(1)}\, d\tau, \qquad t \in [0, \infty), \qquad (4.31)$$

which has the same form as the reciprocity principle associated with the heat equation so that this type of classical reciprocity principle will also hold in the case of a viscous, heat-conducting gas in the absence of mass sources.

5. The initial-boundary value problems for (s, η) and q. In the case of wave propagation in an inviscid, non-conducting gas the problem consists of solving the classical wave equation for a single potential function (usually the velocity potential function or s) subject to suitable initial and boundary conditions. The velocity distribution is then obtained directly from the potential function. When viscous and heat conduction effects are taken into account, however, the problem associated with the classical wave equation is replaced by an initial-boundary value problem which involves both s and η. The velocity distribution is then found by solving an initial-boundary value problem for $\mathbf{q}$.

The equations governing s and η are obtained by using Eqs. (2.4) and (2.5) to eliminate $\mathbf{q}$. We find

$$\frac{\partial \eta}{\partial t} - (\gamma - 1)\frac{\partial s}{\partial t} = \nu' \Delta \eta + g, \qquad (x, t) \in D \times (0, \infty), \tag{5.1}$$

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = p_0 \Delta(s + \eta) + \frac{4}{3}\mu \Delta \frac{\partial s}{dt} + F, \qquad (x, t) \in D \times (0, \infty), \tag{5.2}$$

where $F \equiv \rho_0(\partial f/\partial t) - \frac{4}{3}\mu \Delta f$. The initial and boundary conditions on η will again be given by (2.8) and (2.10)–(2.12) respectively. The initial condition on s is given by (2.7) and we impose an appropriate boundary condition on s of the form

$$s = S(x, t), \qquad (x, t) \in \partial D \times (0, \infty), \tag{5.3}$$

where S is prescribed. An initial condition can be imposed on $\partial s/\partial t$ by using (2.1). This is

$$\frac{\partial s}{\partial t}(x, 0) = l_0, \qquad x \in D, \tag{5.4}$$

where $l_0 \equiv f(x, 0) - \nabla \cdot \mathbf{q}_0$.

The subsequent initial-boundary value problem for $\mathbf{q}$ has the form

$$\rho_0 \frac{\partial \mathbf{q}}{\partial t} = \mu \Delta \mathbf{q} + \mathbf{H}(x, t), \qquad (x, t) \in D \times (0, \infty), \tag{5.5}$$

$$\mathbf{q}(x, 0) = \mathbf{q}_0(x), \qquad x \in D, \tag{5.6}$$

$$\mathbf{q} = \mathbf{Q}, \qquad (x, t) \in \partial D \times (0, \infty), \tag{5.7}$$

where $\mathbf{H}(x, t)$ (prescribed by the solution of the initial-boundary value problem for (s, η)) is given by

$$\mathbf{H} = -p_0 \nabla(s + \eta) + \frac{\mu}{3}\nabla\left(f - \frac{\partial s}{\partial t}\right). \tag{5.8}$$

In the following two sections we will establish variational and reciprocity principles associated with the above initial-boundary value problems for (s, η) and $\mathbf{q}$.

6. Variational principles. To characterize the initial-boundary value problem for (s, η) by a variational principle we replace it, as in Sec. 3, by an equivalent boundary-value problem. This will consist of the field equations

$$\eta - \eta_0 - (\gamma - 1)(s - s_0) = \nu' 1 * (\Delta \eta) + 1 * g, \qquad (x, t) \in D \times (0, \infty), \tag{6.1}$$

$$\rho_0(s - s_0 - l_0 t) = p_0 t^*(\Delta(s + \eta)) + \frac{4}{3}\mu 1^*(\Delta s)$$
$$- \frac{4}{3}\mu(\Delta s_0)t + t^*F, \qquad (x, t) \in D \times (0, \infty) \qquad (6.2)$$

together with the boundary conditions (2.10)–(2.12) and (5.3). It is convenient in what follows to replace Eqs. (6.1) and (6.2) by the equivalent pair of equations

$$1^*(\eta - (\gamma - 1)s) - \nu' t^*(\Delta\eta) = G_1, \qquad (x, t) \in D \times (0, \infty), \qquad (6.3)$$

$$\rho_0 s + \frac{(\gamma - 1)p_0}{\nu'} 1^* s - \frac{p_0 1^* \eta}{\nu'} - p_0 t^*(\Delta s)$$
$$- \frac{4}{3}\mu 1^*(\Delta s) = G_2, \qquad (x, t) \in D \times (0, \infty), \qquad (6.4)$$

where

$$G_1 = t^* g + (\eta_0 - (\gamma - 1)s_0)t, \qquad (6.5)$$

$$G_2 = \rho_0(s_0 + l_0 t) - \frac{4}{3}\mu(\Delta s_0)t + t^*F - \frac{p_0}{\nu'}\eta_0 t + \frac{(\gamma - 1)}{\nu'} p_0 s_0 t - \frac{p_0}{\nu'} t^* g. \qquad (6.6)$$

The variational principle associated with the above system of equations can be stated in the form: Let s, η and their spatial derivatives belong to a function space M. For each $t \in [0, \infty)$ define the functional $V_t(s, \eta)$ on M by the relation

$$V_t(s, \eta) = \frac{1}{2}\nu' \int_D t^*(\nabla\eta)^* \cdot (\nabla\eta)\, d\tau + \frac{1}{2}\int_D 1^*\eta^*\eta\, d\tau$$
$$- (\gamma - 1)\int_D 1^* s^* \eta\, d\tau - \int_D G_1{}^*\eta\, d\tau + \frac{1}{2}\frac{\rho_0 \nu'(\gamma - 1)}{p_0}\int_D s^* s\, d\tau$$
$$+ \frac{1}{2}(\gamma - 1)^2 \int_D 1^* s^* s\, d\tau - \frac{\nu'(\gamma - 1)}{p_0}\int_D G_2{}^* s\, d\tau + \frac{1}{2}\nu'(\gamma' - 1)\int_D t^*(\nabla s)^* \cdot (\nabla s)\, d\tau$$
$$+ \frac{2}{3}\frac{\mu\nu'(\gamma - 1)}{p_0}\int_D 1^*(\nabla s)^* \cdot (\nabla s)\, d\tau - \nu'(\gamma - 1)\int_{\partial D} t^*(s - S)^* \frac{\partial s}{\partial \nu}\, d\sigma$$
$$- \frac{4}{3}\frac{\mu\nu'(\gamma - 1)}{p_0}\int_{\partial D} 1^*(s - S)^* \frac{\partial s}{\partial \nu}\, d\sigma - \nu' \int_{\partial D_1} t^* \frac{\partial \eta^*}{\partial \nu}(\eta - f_1)\, d\sigma - \nu' \int_{\partial D_2} t^* f_2{}^* \eta\, d\sigma$$
$$+ \nu' \int_{\partial D_3} t^*(\tfrac{1}{2}k\eta^*\eta - f_3{}^*\eta)\, d\sigma, \qquad t \in [0, \infty); \qquad (6.7)$$

then

$$\delta V_t(s, \eta) = 0 \text{ on } M, \qquad t \in [0, \infty), \qquad (6.8)$$

iff (s, η) is a solution of the initial-boundary value problem (2.7), (2.8), (2.10)–(2.12) and (5.1)–(5.4).

Proof: We proceed in the same manner as in Sec. 3 to find, after using the divergence theorem,

$$\delta V_t(s, \eta) = \int_D (1*(\eta - (\gamma - 1)s) - \nu' t*(\Delta\eta) - G_1)* \, \delta\eta \, d\tau$$
$$+ \frac{(\gamma - 1)\nu'}{p_0} \int_D \left(\rho_0 s + \frac{(\gamma - 1)}{\nu'} p_0 1*s - G_2 - \frac{p_0 1*\eta}{\nu'} - p_0 t*(\Delta s) - \frac{4}{3}\mu 1*(\Delta s)\right)^* \delta s \, d\tau$$
$$- \nu'(\gamma - 1) \int_{\partial D} t*(s - S)*\left(\frac{\partial}{\partial\nu}\delta s\right) d\sigma - \frac{4\mu\nu'(\gamma - 1)}{3p_0} \int_{\partial D} 1*(s - S)*\left(\frac{\partial}{\partial\nu}\delta s\right) d\sigma$$
$$- \nu' \int_{\partial D_1} t*(\eta - f_1)*\left(\frac{\partial}{\partial\nu}\delta\eta\right) d\sigma + \nu' \int_{\partial D_2} t*\left(\frac{\partial\eta}{\partial\nu} - f_2\right)^* \delta\eta \, d\sigma$$
$$+ \nu' \int_{\partial D_3} t*\left(\frac{\partial\eta}{\partial\nu} + k\eta - f_3\right)^* \delta\eta \, d\sigma, \qquad t \in [0, \infty). \tag{6.9}$$

The rest of the proof follows along similar lines as that in Sec. 3. By the appropriate choice of δs, $\delta\eta$ in $\bar{D} \times [0, \infty)$ and $(\partial/\partial\nu)\delta s$, $(\partial/\partial\nu)\delta\eta$ on $\partial D \times [0, \infty)$ we can, using the fundamental lemma of the variational calculus [1], recover all the equations characterizing the initial-boundary value problem for (s, η). For brevity the details will be omitted.

As in Sec. 3, the above variational principle can be simplified by the method discussed in [6] with the introduction of the functional $\bar{V}_t$ where

$$t*\bar{V}_t \equiv V_t \tag{6.10}$$

The variational principle associated with $\bar{V}_t$ is the same as (6.8) with V_t replaced by $\bar{V}_t$. The functional $\bar{V}_t$ is obtained by differentiating (6.10) twice with respect to t. We find

$$\bar{V}_t = \tfrac{1}{2}\nu' \int_D (\nabla\eta)*\cdot(\nabla\eta) \, d\tau + \tfrac{1}{2} \int_D (\eta(x, 0)\eta(x, t) + \eta*\dot{\eta}) \, d\tau$$
$$- (\gamma - 1) \int_D (s(x, 0)\eta(x, t) + \dot{s}*\eta) \, d\tau - \int_D ((\eta_0 - (\gamma - 1)s_0)\eta(x, t) + g*\eta) \, d\tau$$
$$+ \frac{1}{2} \frac{\rho_0\nu'(\gamma - 1)}{p_0} \int_D (2s(x, 0)\dot{s}(x, t) + \dot{s}*\dot{s}) \, d\tau$$
$$+ \tfrac{1}{2}(\gamma - 1)^2 \int_D (s(x, 0)s(x, t) + s*\dot{s}) \, d\tau$$
$$- \frac{\nu'(\gamma - 1)}{p_0} \int_D \left(\rho_0 s_0 \dot{s} + F*s - \frac{p_0}{\nu 1} g*s\right.$$
$$\left. + \left(l_0 - \frac{4}{3}\mu\Delta s_0 - \frac{p_0\eta_0}{\nu'} - \frac{(\gamma - 1)}{\nu'} p_0 s_0\right) s(x, t)\right) d\tau$$
$$+ \tfrac{1}{2}\nu'(\gamma - 1) \int_D (\nabla s)*\cdot(\nabla s) \, d\tau$$
$$+ \frac{2}{3} \frac{\mu\nu\,(\gamma - 1)}{p_0} \int_D ((\nabla s(x, 0))\cdot(\nabla s(x, t)) + (\nabla s)*\cdot(\nabla\dot{s})) \, d\tau$$
$$- \nu'(\gamma - 1) \int_{\partial D} (s - S)* \frac{\partial s}{\partial\nu} \, d\sigma - \frac{4}{3} \frac{\mu\nu'(\gamma - 1)}{p_0} \int_{\partial D} \left((s(x, 0) - S(x, 0)) \frac{\partial s(x, t)}{\partial\nu}\right.$$
$$\left. + (\dot{s} - \dot{S})* \frac{\partial s}{\partial\nu}\right) d\sigma - \nu' \int_{\partial D_1} \frac{\partial\eta^*}{\partial\nu} (\eta - f_1) \, d\sigma - \nu' \int_{\partial D_2} f_2*\eta \, d\sigma$$
$$+ \nu' \int_{\partial D_3} (\tfrac{1}{2}k\eta*\eta - f_3*\eta) \, d\sigma, \qquad t \in [0, \infty). \tag{6.11}$$

Next we establish a variational principle which characterizes the initial-boundary value problem (5.5)–(5.7) for $\mathbf{q}$. This initial-boundary value problem is similar to the heat conduction problems studied in [1], [5] and [6] and the appropriate Gurtin-type functional required for a variational formulation is $W_t(\mathbf{q})$ where

$$W_t(\mathbf{q}) = \tfrac{1}{2}\mu \sum_{i=1}^{n} \int_D 1*(\nabla u_i)*\cdot(\nabla u_i)\, d\tau + \tfrac{1}{2}\rho_0 \int_D (\mathbf{q} - 2\mathbf{q}_0)*\cdot\mathbf{q}\, d\tau - \int_D \mathbf{H}*\cdot\mathbf{q}\, d\tau - \mu \int_{\partial D} 1*(\mathbf{q} - \mathbf{Q})*\cdot\frac{\partial\mathbf{q}}{\partial\nu}\, d\sigma, \qquad t \in [0, \infty). \tag{6.12}$$

The functional $W_t(\mathbf{q})$ is defined on some function space N and the variational principle can be stated in the form:

$$\delta W_t(\mathbf{q}) = 0 \text{ on } N, \qquad t \in [0, \infty). \tag{6.13}$$

iff $\mathbf{q}$ is a solution of Eqs. (5.5)–(5.7). The simplified functional $\bar{W}_t(\mathbf{q})$ corresponding to $W_t(\mathbf{q})$ is given by

$$1*\bar{W}_t(\mathbf{q}) \equiv W_t\,, \tag{6.14}$$

so that on differentiating (6.14) with respect to time we find

$$\bar{W}_t(\mathbf{q}) = \tfrac{1}{2}\mu \sum_{i=1}^{n} \int_D (\nabla u_i)*\cdot(\nabla u_i) d\tau + \tfrac{1}{2}\rho_0 \int_D (\mathbf{q}(x, 0)\cdot\mathbf{q}(x, t) + \mathbf{q}*\cdot\dot{\mathbf{q}} - 2\mathbf{q}_0\cdot\mathbf{q}(x, t)) d\tau - \int_D (\mathbf{H}(x, 0)\cdot\mathbf{q}(x, t) + \dot{H}*\cdot\mathbf{q}) d\tau - \mu \int_{\partial D} (\mathbf{q} - \mathbf{Q})*\cdot\frac{\partial\mathbf{q}}{\partial\nu}\, d\sigma, \qquad t \in [0, \infty). \tag{6.15}$$

The proof of this latter variational principle (6.13) or the equivalent variational principle associated with $\bar{W}_t(\mathbf{q})$ can be carried out in the same manner as that in Sec. 3.

7. Reciprocity principles. To derive a reciprocity principle associated with the initial-boundary value problem for (s, η) we proceed as in Sec. 4 and set $s = s' + s''$, $\eta = \eta' + \eta''$ where (s'', η'') satisfies the equations

$$\frac{\partial\eta''}{\partial t} - (\gamma - 1)\frac{\partial s''}{\partial t} = \nu'\Delta\eta'', \qquad (x, t) \in D \times (0, \infty), \tag{7.1}$$

$$\rho_0 \frac{\partial^2 s''}{\partial t^2} = p_0\Delta(\eta'' + s'') + \frac{4}{3}\mu\Delta\frac{\partial s''}{\partial t}, \qquad (x, t) \in D \times (0, \infty), \tag{7.2}$$

together with the initial and boundary conditions (2.8), (2.10)–(2.12) and (5.3)–(5.4). The functions s' and η' then satisfy the initial-boundary value problem

$$\frac{\partial\eta'}{\partial t} - (\gamma - 1)\frac{\partial s'}{\partial t} = \nu'\Delta\eta' + g, \qquad (x, t) \in D \times (0, \infty), \tag{7.3}$$

$$\rho_0 \frac{\partial^2 s'}{\partial t^2} = p_0\Delta(\eta' + s') + \frac{4}{3}\mu\Delta\frac{\partial s'}{\partial t} + F, \qquad (x, t) \in D \times (0, \infty), \tag{7.4}$$

$$s'(x, 0) = 0, \qquad x \in D, \tag{7.5}$$

$$\eta'(x, 0) = 0, \qquad x \in D, \tag{7.6}$$

$$s' = 0, \qquad (x, t) \in \partial D \times (0, \infty), \tag{7.7}$$

$$\eta' = 0, \qquad (x, t) \in \partial D_1 \times (0, \infty), \tag{7.8}$$

$$\frac{\partial \eta'}{\partial \nu} = 0, \qquad (x, t) \in \partial D_2 \times (0, \infty), \tag{7.9}$$

$$\frac{\partial \eta'}{\partial \nu} + k(x)\eta' = 0, \qquad (x, t) \in \partial D_3 \times (0, \infty). \tag{7.10}$$

In the appendix a uniqueness proof is given for the initial-boundary value problem for (s, η) so that the reciprocity principle can be stated in the form: if (s_i', η_i'), $i = 1, 2$ are the solutions of (7.3)–(7.10) corresponding to the source pair $(F^{(1)}, g^{(1)})$ and $(F^{(2)}, g^{(2)})$ respectively, then

$$\int_D \left\{s_1'*\left(F^{(2)} - \frac{p_0}{\nu'} g^{(2)}\right) + \frac{p_0}{\nu'} (\gamma - 1)^{-1}\eta_1'*g^{(2)}\right\} d\tau$$
$$= \int_D \left\{s_2'*\left(F^{(1)} - \frac{p_0}{\nu'} g^{(1)}\right) + \frac{p_0}{\nu'} (\gamma - 1)^{-1}\eta_2'*g^{(1)}\right\} d\tau, \qquad t \in [0, \infty). \tag{7.11}$$

Proof: The system of equations equivalent to (7.3)–(7.6) is

$$\eta' - (\gamma - 1)s' = \nu' 1*(\Delta\eta') + 1*g, \qquad (x, t) \in D \times (0, \infty), \tag{7.12}$$

$$\rho_0 s' = p_0 t*(\Delta(\eta' + s')) + \frac{4}{3}\mu 1*(\Delta s') + t*F, \qquad (x, t) \in D \times (0, \infty). \tag{7.13}$$

Using the boundary condition (7.7) together with the divergence theorem we obtain the relation

$$\int_D \left\{p_0 t*(s_2'*(\Delta s_1') - s_1'*(\Delta s_2')) + \frac{4}{3}\mu 1*(s_2'*(\Delta s_1') - s_1'*(\Delta s_2'))\right\} d\tau = 0. \tag{7.14}$$

From (7.12) and (7.13) we have

$$\rho_0 s' = p_0 t*(\Delta s') + \frac{4}{3}\mu 1*(\Delta s') + t*F + \frac{p_0}{\nu'} 1*(\eta' - (\gamma - 1)s') - \frac{p_0}{\nu'} t*g,$$
$$(x, t) \in D \times (0, \infty), \tag{7.15}$$

so that using (7.14) and differentiating with respect to time we obtain

$$\int_D \left\{1*(s_1'*F^{(2)} - s_2'*F^{(1)}) - \frac{p_0}{\nu'} 1*(s_1'*g^{(2)} - s_2'*g^{(1)}) \right.$$
$$\left. + \frac{p_0}{\nu'} (s_1'*\eta_2' - s_2'*\eta_1')\right\} d\tau = 0. \tag{7.16}$$

Again, using the boundary conditions (7.8)–(7.10), Eq. (7.12) and the divergence theorem, we obtain the relation

$$(\gamma - 1)\int_D (\eta_2'*s_1' - \eta_1'*s_2')\, d\tau = \int_D 1*(\eta_1'*g^{(2)} - \eta_2'*g^{(1)})\, d\tau. \tag{7.17}$$

Combining (7.16) and (7.17) and differentiating with respect to time, we find

$$\int_D \left\{s_1'*\left(F^{(2)} - \frac{p_0}{\nu'} g^{(2)}\right) + \frac{p_0}{\nu'} (\gamma - 1)^{-1}\eta_1'*g^{(2)}\right\} d\tau$$
$$= \int_D \left\{s_2'*\left(F^{(1)} - \frac{p_0}{\nu'} g^{(1)}\right) + \frac{p_0}{\nu'} (\gamma - 1)^{-1}\eta_2'*g^{(1)}\right\} d\tau, \qquad t \in [0, \infty). \tag{7.18}$$

As in Sec. 4, we note special cases of the reciprocity principle (7.18). *If heat sources are absent* then $g^{(i)} = 0$, $i = 1, 2$ and the reciprocity principle has the form

$$\int_D s_1'{*}F^{(2)}\,d\tau = \int_D s_2'{*}F^{(1)}\,d\tau, \qquad t \in [0, \infty), \tag{7.19}$$

which is similar to the reciprocity principle (associated with fractional density changes) obtained in [4] for the classical wave equation. The source function, however, in this case is $\rho_0(\partial f/\partial t) - \frac{4}{3}\mu\Delta f$. If, further, viscous effects are neglected then $F = \rho_0(\partial f/\partial t)$, Eq. (7.4) is replaced by the classical equation

$$\rho_0 \frac{\partial^2 s'}{\partial t^2} = p_0\,\Delta s' + \rho_0 \frac{\partial f}{\partial t}, \qquad (x, t) \in D \times (0, \infty), \tag{7.20}$$

and the reciprocity principle reduces to

$$\int_D s_1'{*}f^{(2)}\,d\tau = \int_D s_2'{*}f^{(1)}\,d\tau, \qquad t \in [0, \infty), \tag{7.21}$$

which is the same as that obtained in [4]. Again, if the source functions f and g are chosen such that $\rho_0(\partial f/\partial t) - \frac{4}{3}\mu\Delta f = (p_0/\nu')g$ then the reciprocity principle (7.18) can be written

$$\int_D \eta_1'{*}g^{(2)}\,d\tau = \int_D \eta_2'{*}g^{(1)}\,d\tau, \qquad t \in [0, \infty). \tag{7.22}$$

This latter statement of the reciprocity principle has the same form as Eq. (4.31) which was obtained for the general problem with $f = 0$. Also, if the source functions f and g are chosen such that $F = \gamma/(\gamma - 1)(p_0/\nu')g$ then the reciprocity principle (7.16) has the form

$$\int_D (s_1' + \eta_1'){*}g^{(2)}\,d\tau = \int_D (s_2' + \eta_2'){*}g^{(1)}\,d\tau, \qquad t \in [0, \infty). \tag{7.23}$$

or

$$\int_D (s_1' + \eta_1'){*}F^{(2)}\,d\tau = \int_D (s_2' + \eta_2'){*}F^{(1)}\,d\tau, \qquad t \in [0, \infty). \tag{7.24}$$

The forms of the reciprocity principle given by Eqs. (7.23) and (7.24) are similar to that given by (4.30).

Finally, we will derive a reciprocity principle which is associated with the initial-boundary value problem for $\mathbf{q}$. We set $\mathbf{q} = \mathbf{q}' + \mathbf{q}''$ where $\mathbf{q}''$ satisfies

$$\rho_0(\partial \mathbf{q}''/\partial t) = \mu\Delta\mathbf{q}'', \qquad (x, t) \in D \times (0, \infty), \tag{7.25}$$

together with the initial and boundary conditions (5.6) and (5.7). The initial-boundary value problem for $\mathbf{q}'$ can be written as n initial-boundary value problems for u_i', $i = 1, \cdots n$. These are

$$\rho_0(\partial u_i'/\partial t) = \mu\Delta u_i' + H_i, \qquad (x, t) \in D \times (0, \infty), \qquad i = 1, \cdots n, \tag{7.26}$$

$$u_i'(x, 0) = 0, \qquad x \in D, \qquad i = 1, \cdots n, \tag{7.27}$$

$$u_i' = 0, \qquad (x, t) \in \partial D \times (0, \infty), \qquad i = 1, \cdots n, \tag{7.28}$$

where $\mathbf{H} = (H_1, \cdots H_n)$. As before the reciprocity principles can then be stated in the form: if $u_i'^{(j)}$, $i = 1, \cdots n$, are the solutions of the initial-boundary value problems (7.26)–(7.28) associated with the source functions $H_i^{(j)}$, $i = 1, \cdots n$, $j = 1, 2$ then

$$\int_D u_i'^{(1)} * H_i^{(2)} \, d\tau = \int_D u_i'^{(2)} * H_i^{(1)} \, d\tau, \qquad t \in [0, \infty), \qquad i = 1, \cdots, n. \tag{7.29}$$

Proof: The system of equations equivalent to (7.26) and (7.27) is

$$\rho_0 u_i = \mu 1 * (\Delta u_i) + 1 * H_i, \qquad (x, t) \in D \times (0, \infty), \qquad i = 1, \cdots n. \tag{7.30}$$

By using the divergence theorem together with the boundary condition (7.28) and differentiating with respect to time we can readily obtain (7.29). For brevity the details can be omitted.

Appendix. Uniqueness. The solution of the initial-boundary value problem for (s, η) is unique if $s \equiv \eta \equiv 0$, $(x, t) \in \bar{D} \times [0, \infty)$ is the solution of the initial-boundary value problem obtained by setting $g, F, S, l_0, s_0, \eta_0, f_i$, $i = 1, 2$ equal to zero. To obtain this result we use the positive definite function $I(t)$ where

$$I(t) = \tfrac{1}{2}\nu' \int_D \{(\gamma - 1)(\rho_0 s_t^2 + p_0(\nabla s)\cdot(\nabla s)) + p_0(\nabla \eta)\cdot(\nabla \eta)\} \, d\tau + \tfrac{1}{2} p_0 \nu' \int_{\partial D_3} k \eta^2 \, d\sigma, \qquad t \in [0, \infty). \tag{A.1}$$

Differentiating (A.1) with respect to time and using the divergence theorem together with the boundary conditions and two field equations, we find

$$\frac{dI}{dt} = \nu' \int_D (\gamma - 1) s_t \left(\frac{4\mu}{3} \Delta s_t + p_0 \, \Delta \eta \right) d\tau - \int_D p_0 \eta_t (\eta_t - (\gamma - 1) s_t) \, d\tau. \tag{A.2}$$

After a further application of the divergence theorem and equations (5.1) and (5.3) to the first term in Eq. (A.2) we obtain

$$\frac{dI}{dt} = -\frac{4\nu'\mu}{3}(\gamma - 1) \int_D (\nabla s_t)\cdot(\nabla s_t) \, d\tau - p_0 \int_D (\eta_t + (\gamma - 1)s_t)^2 \, d\tau \leq 0, \tag{A.3}$$

so that $I(t) = 0$, $t \in [0, \infty)$, which implies $s \equiv \eta \equiv 0$, $(x, t) \in \bar{D} \times [0, \infty)$ and uniqueness is proved.

Uniqueness can be established for the n initial-boundary value problems for $\mathbf{q}$ in a similar manner. In this case we consider the function $J(t)$ where

$$J(t) = \frac{1}{2} \sum_{i=1}^{n} \int_D (\nabla u_i)\cdot(\nabla u_i) \, d\tau, \qquad t \in [0, \infty), \tag{A.4}$$

and use the divergence theorem together with the homogeneous initial and boundary conditions on $\mathbf{q}$ to show that $dJ/dt \leq 0$ and therefore $J(t) = 0$, $t \in [0, \infty)$, from which uniqueness follows.

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