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STABILITY CRITERIA FOR GENERALIZED MATHIEU EQUATIONS*

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Abstract. A class of differential equations that generalizes Mathieu's is treated. Stability criteria for solutions are obtained. The method can be applied to a wider class of Hill equations.

In this paper we will consider the differential equation

$$\left[\frac{d^2}{dx^2} + \frac{c_1 + c_2 \cos 2x}{c_3 + c_4 \cos 2x}\right] y = 0, \qquad c_4^2 < c_3^2. \tag{1}$$

For $c_4 = 0$ it reduces to Mathieu's equation, and for $c_2 = 0$ to one mentioned in a previous note [1]. The remarkable thing is that we will be able to find an easy way to deal with this equation when $c_4 \neq 0$. *The method cannot be applied to the special case* $c_4 = 0$.

It will prove convenient to write (1) in the form

$$\left[\frac{d^2}{dx^2} + 4P + \frac{4R}{1 - A \cos 2x}\right] y = 0, \qquad A^2 < 1, \tag{2}$$

$$P = c_2/4c_4\,, \quad R = \tfrac{1}{4}(c_1/c_3 - c_2/c_4), \quad A = -c_4/c_3.$$

If we now use the Fourier expansion of (1-Acos $2x)^{-1}$ we obtain the Hill form of our equation:

$$\left[\frac{d^2}{dx^2} + 4P + 4R\left((1 - A^2)^{-1/2} + 2(1 - A^2)^{-1/2} \sum_{n=1}^{\infty} \alpha^n \cos 2nx\right)\right] y = 0, \tag{3}$$

$$\alpha = [1 - (1 - A^2)^{1/2}]/A.$$

Note that each consecutive Fourier amplitude is α times the previous one. It is this property of (3) that we intend to utilize when solving the stability problem. The method could, in fact, be used on any differential equation having the abovementioned quality (e.g. $R \to R \sin x$, $R \to R \cos x$, $R \to R \sin 2x$ in (2)).

Following McLachlan [2], the solutions of (3) can be shown to be periodic functions of x (with period π) multiplied by exp (μx), where

$$\sin^2 (\tfrac{1}{2} i\mu\pi) = \Delta(0) \sin^2 (\pi[P + R/(1 - A^2)^{1/2}]^{1/2}) \tag{4}$$

and $\Delta(0)$ is an infinite determinant. If we denote its elements by a_{mn} and have m go from $-\infty$ to $+\infty$ from the bottom to the top of the page, and n from left to right, then

$$\begin{aligned} a_{mn} &= 1 \quad \text{if} \quad m + n = 0, \\ a_{mn} &= \alpha^{|m+n|}\beta_m\,, \qquad m + n \neq 0, \qquad \beta_m = R/[R - (m^2 - P)(1 - A^2)^{1/2}]. \end{aligned} \tag{5}$$

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Once we have calculated $\Delta(0)$ we will be able to see whether our equation yields unstable or doubly periodic solutions (μ real or pure imaginary). By inverting (4) we can find growth rates and periods of solutions.

If for n positive we now multiply the nth column by α and subtract it from the $n+1$st, the whole upper-right-hand quarter of our determinant becomes zero. Upon completing similar subtractions on the left we see that

$$|a_{mn}| = \begin{vmatrix} SQRT(a) & & 0 \\ 0 \cdots\cdots 0 & 1 & 0 \cdots\cdots 0 \\ 0 & & SQRT(a) \end{vmatrix}, \tag{6}$$

$$SQRT(a) = \begin{vmatrix} 1-\alpha^2\beta_1, \ \alpha(\beta_1-1), \ 0, \cdots\cdots \\ (\alpha-\alpha^3)\beta_2, \ 1-\alpha^2\beta_2, \ \alpha(\beta_2-1), \ 0, \cdots \\ (\alpha^2-\alpha^4)\beta_3, \ (\alpha-\alpha^3)\beta_3, \ 1-\alpha^2\beta_3, \ \alpha(\beta_3-1), \ 0 \\ \cdots\cdots\cdots\cdots \end{vmatrix}. \tag{7}$$

The semi-infinite determinant that remains to be evaluated will be found as the limit of a sequence of $N \times N$ determinants, known to converge [2]. Each of these finite determinants can be further simplified by subtracting columns from right to left and taking out factors. The final form is

$$SQRT(a)_N = \alpha^N \left[\prod_{m=1}^{N} (\beta_m - 1)\right] \begin{vmatrix} \gamma_1, \ 1, \ 0, \ \cdots\cdots 0 \\ 1, \ \gamma_2, \ 1, \ 0 \cdots\cdots 0 \\ 0, \ 1, \ \gamma_3, \ 1, \ 0 \cdots 0 \\ \cdots\cdots\cdots \\ 0, \cdots\cdots 1, \ \gamma_N \end{vmatrix} \tag{8}$$

$$\gamma_m = \frac{1+\alpha^2-2\alpha^2\beta_m}{\alpha(\beta_m-1)}, \qquad m < N; \qquad \gamma_N = \frac{1-\alpha^2}{\alpha(\beta_m-1)}.$$

So finally

$$\sqrt{\Delta(0)} = \lim_{N\to\infty} \alpha^N D_N \prod_{1}^{N} \frac{(m^2-P)\sqrt{1-A^2}}{R-(m^2-P)\sqrt{1-A^2}}. \tag{9}$$

When (9) is substituted into (4) and the product formula for sin x used [3], we obtain

$$\sin\left(\tfrac{1}{2}i\mu\pi\right) = \pm\sqrt{1 + R/(P\sqrt{1 - A^2})}\,\sin(\pi\sqrt{P})\lim_{N\to\infty}\alpha^N D_N\ ,\ P \neq 0$$

$$= \pm\pi\sqrt{R/\sqrt{1 - A^2}}\lim_{N\to\infty}\alpha^N D_N\ ,\ P = 0 \qquad (10)$$

$$D_1 = \gamma_1\ ,\quad D_2 = \gamma_1\gamma_2 - 1,\quad D_k = \gamma_k D_{k-1} - D_{k-2}$$

and this, in principle, solves the problem. Notice how the $\sin^2$ term in (4) is cancelled by a factor of the determinant.

A general stability chart obtainable from (10) would be three-dimensional with P and R general real numbers and $0 \leq A \leq 1$ (the equation corresponding to $-A$ can be obtained from the A equation via the transformation $x \to x + \pi/2$). The stability chart for $P = 0$ is given in Fig. 1. It is interesting to compare it to that of the Mathieu equation. One might expect the two to at least be similar for small A. However, this is not the case. The $P = 0$ chart shows only half as many instability regions as Mathieu's. It would be easy to quote papers in the literature that use the Mathieu stability chart for somewhat different equations. A comparison of Fig. 1 with the Mathieu stability chart [2] illustrates how dangerous this is, even for small A.

Fig. 2 plots the modulus of the right-hand side (RHS) of (10) for $P = 0$, $R = 1$ as a function of A. As R is real, μ can only be real and lead to instability when RHS > 1. This happens in two regions shown in the drawing: $0.9627 < A < 0.9660$ and $A \simeq 0.99971$. In the first region the maximum value of the RHS is 1.00037. This particular case of (2) arose in a physical context [4]. The instability regions and growth rates are so small that for most practical purposes we conclude stability for $(P, R) = (0, 1)$.

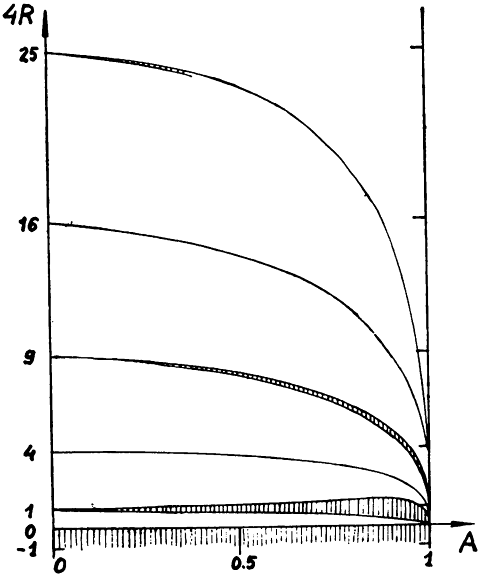

FIG. 1. Stability chart for $P = 0$. Unstable regions are shaded. Curves correspond to singly periodic solutions of (2).

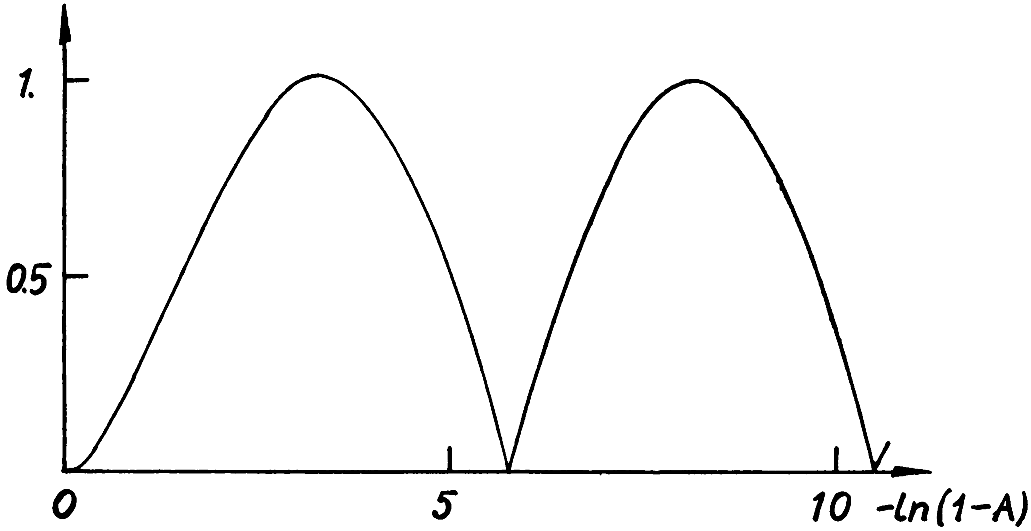

FIG. 2. |RHS| of Eq. (10) as a function of ln(1 − A). Instability occurs when |RHS| > 1. The first two unstable regions are indicated.

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