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ON THE LINEAR CAPILLARY GRAVITY WAVE PROBLEM*

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Summary. A general initial-boundary value problem is formulated for capillary-gravity waves which includes the derivation of a boundary condition at the liquid-solid intersection. Conditions on the bounding surface geometry which ensure uniqueness are also established.

1. Introduction. Among the many surface tension-driven phenomena studied in the literature [1], few attempts have been made to solve the linear capillary-gravity wave problem when a solid intersects the free surface because the additional forces due to surface tension are relatively small. Apart from the added term that appears in the dynamic free surface condition, the main difficulty lies in deciding what the boundary condition at a liquid-solid intersection should be. In the recent review [2] this condition has not been formulated. Furthermore, in the investigations [3], [4] and [5] the need for a boundary condition to ensure uniqueness has been recognized but the boundary condition employed required the introduction of singularities in the velocity potential. The free surface slope at a liquid-solid intersection was specified and though the form of the condition was correct in [3] and [4], the condition of irrotationality was violated at the liquid-solid intersection. Moreover, in [5] no allowance was made for the local geometry of the solid boundary at liquid-solid intersections.

The purpose of this paper is to derive the boundary condition for a general liquid-solid intersection using the conditions of irrotationality, the kinematic free surface condition and the equation of continuity together with the boundary condition which arises naturally on the solid surfaces bounding the fluid. In Sec. 2 a general initial-boundary value problem is formulated which includes the required boundary condition and in Sec. 3 a uniqueness theorem is established under certain conditions imposed on the geometry of the bounding surfaces.

2. Problem formulation. We consider the irrotational motion of an incompressible fluid in a domain bounded by the surfaces $\Sigma \cup \Sigma'$ and a free surface which remains in the vicinity of Γ throughout the motion (see Fig. 1). The surface Γ lies in the plane $z = 0$, D is the domain bounded by $\Sigma \cup \Gamma$ and C represents the curve of intersection of Γ and Σ. A point in D is denoted by $\mathbf{r} = (x, y, z)$ and the z axis is vertically upwards. The fluid motion is characterized by a velocity potential ϕ and free surface displacement ζ which satisfy the system of equations

$$\nabla^2\phi = 0 \quad \text{in} \quad D, t \geq 0, \tag{2.1}$$

$$\phi(\mathbf{r}, 0) = f_1(\mathbf{r}), \zeta(\mathbf{r}, 0) = f_2(\mathbf{r}) \quad \text{on} \quad \Gamma, t \geq 0, \tag{2.2}$$

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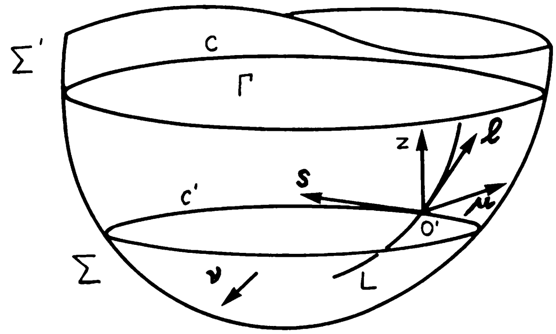

$$\partial\phi/\partial\nu = U(\mathbf{r}, t) \quad \text{on} \quad \Sigma, t \geq 0, \tag{2.3}$$

$$\partial\zeta/\partial t = \partial\phi/\partial z \quad \text{on} \quad \Gamma, t \geq 0, \tag{2.4}$$

$$\rho \frac{\partial\phi}{\partial t} + \rho g\zeta - T\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right) = P(\mathbf{r}, t) \quad \text{on} \quad \Gamma, \qquad t \geq 0, \tag{2.5}$$

where ρ is the fluid density, g the gravitational acceleration, and T the tension acting over the free surface. The functions f_1 and f_2 are the initial values of $\phi(x, y, 0, t)$ and $\zeta(x, y, t)$ respectively, P is the prescribed pressure over the free surface and U is the prescribed velocity normal to Σ with the outward normal to Σ denoted by $\boldsymbol{\nu}$.

To derive the boundary condition on C we employ the orthogonal curvilinear coordinate system (O', μ, s, z) on the curve C' which is the intersection of the plane $z = -d$, $d > 0$ with Σ. In this system $\boldsymbol{\mu}$ is the outward normal to C' that lies in the plane $z = -d$ and $\mathbf{s}$ is tangential to C'. If q_μ, q_s and w denote the velocity components of the fluid at O' then the equation of continuity and the conditions of irrotationality can be written in the form

$$\frac{\partial q_\mu}{\partial\mu} + \frac{\partial q_s}{\partial s} + \frac{\partial w}{\partial z} = 0, \tag{2.6}$$

$$\frac{\partial q_\mu}{\partial z} - \frac{\partial w}{\partial\mu} = 0, \tag{2.7}$$

$$\frac{\partial q_s}{\partial z} - \frac{\partial w}{\partial s} = 0, \tag{2.8}$$

$$\frac{\partial q_\mu}{\partial s} - \frac{\partial q_s}{\partial\mu} = 0. \tag{2.9}$$

Furthermore, if a plane normal to C' intersects Σ in a curve L then the boundary condition (2.3) may be written

$$q_\mu \sin\alpha - w\cos\alpha = U, \tag{2.10}$$

where α is the angle between $\boldsymbol{\mu}$ and $\mathbf{l}$, the tangent to L.

Differentiating (2.10) with respect to μ and using (2.6), we obtain

$$-\left(\frac{\partial q_s}{\partial s} + \frac{\partial w}{\partial z}\right)\sin\alpha + (q_\mu\cos\alpha + w\sin\alpha)\frac{\partial\alpha}{\partial\mu} - \cos\alpha\frac{\partial w}{\partial\mu} = \frac{\partial U}{\partial\mu}. \tag{2.11}$$

Also, from (2.7) and (2.10) we obtain

$$\begin{aligned}\frac{\partial w}{\partial z} &= \frac{\partial}{\partial z}(q_\mu \tan\alpha - U\sec\alpha) \\ &= \tan\alpha\frac{\partial q_\mu}{\partial z} + q_\mu \sec^2\alpha\frac{\partial \alpha}{\partial z} - \frac{\partial}{\partial z}(U\sec\alpha) \\ &= \tan\alpha\frac{\partial w}{\partial \mu} - \sec\alpha\frac{\partial U}{\partial z} + (w\cot\alpha\sec^2\alpha + U\operatorname{cosec}\alpha)\frac{\partial\alpha}{\partial z}. \qquad (2.12)\end{aligned}$$

Combining (2.10)–(2.12) and using the identity

$$\frac{\partial}{\partial l} = -\cos\alpha\frac{\partial}{\partial\mu} + \sin\alpha\frac{\partial}{\partial z}, \qquad (2.13)$$

we find, after some algebra,

$$\frac{\partial w}{\partial \mu} + \kappa\operatorname{cosec}\alpha w = -\tfrac{1}{2}\sin 2\alpha\frac{\partial q_s}{\partial s} + \sin\alpha\frac{\partial}{\partial l}(U\operatorname{cosec}\alpha), \qquad (2.14)$$

where $K \equiv \partial\alpha/\partial \mathrm{l}$, the curvature of L.

In the limit $d \to 0$ we can use (2.4) to replace $w(\equiv \partial\phi/\partial z)$ in (2.14) and obtain a boundary condition on C in the form

$$\frac{\partial^2\zeta}{\partial t\,\partial\mu} + \kappa\operatorname{cosec}\alpha\frac{\partial\zeta}{\partial t} = -\tfrac{1}{2}\sin 2\alpha\frac{\partial q_s}{\partial s} + \sin\alpha\frac{\partial}{\partial l}(U\operatorname{cosec}\alpha). \qquad (2.15)$$

The two-dimensional form of the boundary condition is

$$\frac{\partial^2\zeta}{\partial t\,\partial x} + \kappa\operatorname{cosec}\alpha\frac{\partial\zeta}{\partial t} = \sin\alpha\frac{\partial}{\partial l}(U\operatorname{cosec}\alpha), \qquad (2.16)$$

after replacing μ by x and setting q_s equal to zero.

In what follows we will discuss the conditions under which these boundary conditions together with Eqs. (2.1)–(2.5) and the remaining conditions of irrotationality (2.8)–(2.9) ensure a unique solution to the linear capillary-gravity wave problem.

3. Uniqueness. To prove uniqueness we set f_1, f_2, U and $P \equiv 0$ and show that $\phi = \zeta = 0$ is the solution of the resulting initial-boundary value problem under conditions to be imposed on Σ.

Since $\zeta = \partial\zeta/\partial\mu = 0$, $t = 0$ and $U = 0$, $t \geq 0$ then $\partial\alpha/\partial t = 0$, $t > 0$ and (2.15) can be integrated with respect to t so that

$$\frac{\partial\zeta}{\partial\mu} + \kappa\operatorname{cosec}\alpha\zeta = -\tfrac{1}{2}\sin 2\alpha\int_0^t \frac{\partial q_s}{\partial s}\,d\tau, \qquad t > 0. \qquad (3.1)$$

For $0 < \alpha < \pi$ we define the positive definite function $E(t)$ by the relation

$$E = \frac{1}{2}\iiint_D \rho(\nabla\phi)\cdot(\nabla\phi)\,dV + \iint_\Gamma \rho g\zeta^2\,d\sigma$$
$$+ \frac{1}{2}\iint_C T(\nabla_\Gamma\zeta)\cdot(\nabla_\Gamma\zeta)\,d\sigma + \int_C T\,|\kappa|\operatorname{cosec}\alpha\zeta^2\,ds, \qquad t \geq 0. \qquad (3.2)$$

where $\nabla_\Gamma \equiv (\partial/\partial x, \partial/\partial y)$. By employing the divergence theorem in D and on Γ together with (2.1), (2.3)–(2.5) and (3.1) it can readily be shown that

$$\frac{dE}{dt} = \frac{d}{dt}\int_C T(|\kappa| - \kappa) \operatorname{cosec} \alpha \zeta^2 \, ds - \tfrac{1}{2}T \int_0^t \int_C \sin 2\alpha w \frac{\partial q_s}{\partial s} \, ds \, d\tau, \qquad t > 0. \tag{3.3}$$

Integrating by parts in the second line integral on the right of (3.3) and using the fact that C is a closed curve, we obtain

$$\frac{dE}{dt} - \frac{d}{dt}\int_C T(|\kappa| - \kappa) \operatorname{cosec} \alpha \zeta^2 \, ds = \tfrac{1}{2}T \int_0^t \int_C q_s \frac{\partial}{\partial s}(\sin 2\alpha w) \, ds \, d\tau, \qquad t > 0, \tag{3.4}$$

$$= T \int_0^t \int_C q_s \frac{\partial}{\partial s}(q_\mu \sin^2 \alpha) \, ds \, d\tau, \qquad t > 0, \tag{3.5}$$

by virtue of (2.10). Adding (3.4) and (3.5), we find

$$\frac{dE}{dt} - \frac{d}{dt}\int_C T(|\kappa| - \kappa) \operatorname{cosec} \alpha \zeta^2 \, ds$$

$$= \tfrac{1}{2}T \int_0^t \int_C q_s(w \cos 2\alpha + q_\mu \sin 2\alpha) \frac{\partial \alpha}{\partial s} \, ds \, d\tau$$

$$+ \tfrac{1}{2}T \int_0^t \int_C q_s \sin \alpha \left(\cos \alpha \frac{\partial w}{\partial s} + \sin \alpha \frac{\partial q_\mu}{\partial s}\right) ds \, d\tau, \qquad t > 0. \tag{3.6}$$

By using (2.8)–(2.10) and the identity

$$\frac{\partial}{\partial \nu} = \sin \alpha \frac{\partial}{\partial \mu} + \cos \alpha \frac{\partial}{\partial z} \tag{3.7}$$

we can write (3.6) in the form

$$\frac{dE}{dt} - \frac{d}{dt}\int_C T(|\kappa| - \kappa) \operatorname{cosec} \alpha \zeta^2 \, ds$$

$$= \tfrac{1}{2}T \int_0^t \int_C q_s w(2 \cos 2\alpha + 1) \frac{\partial \alpha}{\partial s} \, ds \, d\tau, \qquad t > 0, \tag{3.8}$$

since $\partial q_s/\partial y = \partial^2\phi/\partial y \, \partial s = 0$.

When $K \geq 0$ and $\partial\alpha/\partial s = 0$ on C we obtain

$$dE/dt = 0, \qquad t > 0, \tag{3.9}$$

which implies $E(t) = 0$, $t \geq 0$ since $E(0) = 0$. We therefore obtain $\phi = \zeta = 0$, $t \geq 0$ and uniqueness is established for a general Σ under the conditions stated above. The corresponding condition in the two dimensional case is that $K \geq 0$ at liquid-solid points of intersection.

In the special case when $\alpha \simeq \pi/2$ so that second-order terms on the right of (2.15) can be neglected, the boundary condition assumes the simpler form

$$\frac{\partial^2 \zeta}{\partial t \, \partial \mu} + K \frac{\partial \zeta}{\partial t} = \frac{\partial U}{\partial l}, \qquad t > 0. \tag{3.10}$$

Uniqueness is ensured in this case again if $K \geq 0$. If, further, K is sufficiently small that the second term on the left of (3.10) can be neglected, then the boundary condition is of the Neumann form and uniqueness is ensured when only the functions f_1, f_2, U and P are prescribed as is the case for the analogous gravity wave problem.

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