UNIVERSAL SOLUTIONS FOR THERMODYNAMIC FOURIER MATERIAL*

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1. Introduction. There has of late been much interest in universal solutions for homogeneous isotropic thermoelastic materials. Petroski and Carlson [1] have shown that there is a certain paucity of universal solutions, that in the case of incompressible material there are only two types of solutions, and that in the compressible case there is none, except for the trivial solution with a constant temperature field.

One reason for this paucity in the incompressible case is the fact that temperature fields must be of the form

$$\theta = k + p\varphi + qz \tag{1}$$

(using cylindrical coordinates (r, φ, z)) according to a theorem by Hamel [2].

In this paper we are concerned with a subclass of thermoelastic materials for which Hamel's theorem is not applicable, and we are able to show that there exist four types of universal solutions for this subclass. In the last section of this paper we consider a compressible material and prove that for suitably chosen subclasses one can find more universal solutions.

2. Thermodynamic preliminaries. We will use the notations generally found in rational thermodynamics (e.g. Truesdell [3]). Let $F = \partial \mathbf{x}/\partial \mathbf{x}$ be the deformation gradient of the deformation $\mathbf{x} = \mathbf{x}(\mathbf{X})$, $c^{-1} = FF^T$ the left Cauchy–Green tensor, T the Cauchy stress tensor, θ the temperature, $\mathbf{g}$ the temperature gradient, $\mathbf{h}$ the heat flux. η the specific entropy and ψ the free energy. According to the principle of equipresence, the constitutive equation of a simple thermodynamic material may be written in the form:

$$\phi = \mathfrak{C}(F^t, \theta^t, \mathbf{g}^t) \tag{2}$$

where ϕ stands for the set $(T, \mathbf{h}, \psi, \eta)$ and F^t, θ^t, $\mathbf{g}^t$ denote the history of F, θ, $\mathbf{g}$, respectively.

If the equation

$$\phi = C(F, \theta, \mathbf{g}) \tag{3}$$

holds, the material is called thermoelastic.

A thermodynamic process has to satisfy the dynamical equations and the first law of thermodynamics, while the second law is used to put restrictions on the material functions. Hence we have the equations:

$$\text{Cauchy 1:} \quad \operatorname{div} T + \rho \mathbf{b} = \rho \ddot{\mathbf{x}}, \tag{4}$$

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$$\text{Cauchy 2:} \quad T = T^T, \tag{5}$$

$$\text{Fourier-Kirchhoff-Neumann:} \quad \rho\dot{\epsilon} = w + \operatorname{div} \mathbf{h} + \rho s \tag{6}$$

where $\mathbf{b}$ is the body force, ϵ the specific energy, w the specific work and s the specific heat supply.

Fourier material differs from general thermoelastic material by the special form of the heat flux

$$\mathbf{h} = K(F, \theta)\mathbf{g} \tag{7}$$

where K is the heat flux tensor and $\mathbf{h}$ is linear in $\mathbf{g}$.

A thermoelastic material is called incompressible homogeneous isotropic and satisfies the second law of thermodynamics if the constitutive equations can be written in the form

$$T = -p\mathbf{1} + 2\rho_0\left(\frac{\partial \psi}{\partial I} c^{-1} - \frac{\partial \psi}{\partial II} c\right) \tag{8}$$

$$\mathbf{h} = (\varphi_0 \mathbf{1} + \varphi_1 c + \varphi_{-1} c^{-1})\mathbf{g} \tag{9}$$

where $I = \operatorname{tr} c^{-1}$, $II = \frac{1}{2}(I^2 - \operatorname{tr} c^{-2})$ and $\psi = \psi(I, II, \theta)$, $\varphi_i = \varphi_i(I, II, \theta, \mathbf{g}\cdot\mathbf{g}, \mathbf{g}\cdot c\mathbf{g}, \mathbf{g}\cdot c^{-1}\mathbf{g})$.

For the incompressible homogeneous isotropic Fourier material we obtain the same form except that

$$\varphi_i = \varphi_i(I, II, \theta). \tag{10}$$

3. Universal solutions. A universal solution for a thermodynamic material consists of a deformation and a temperature field which can be produced in every thermodynamic body by applying surface tractions and heat fluxes alone without the help of specific body forces and heat supplies. Therefore some authors call them controllable solutions. Since we are here dealing with *static* deformations, we have $\dot{\mathbf{x}} = \mathbf{0}$ for controllable solutions. Hayes, Laws and Osborn [9] have previously considered universal *motions* in thermoelasticity.

In mathematical notation, the problem consists in determining c^{-1} and θ so that the following conditions are satisfied without referring to the special form of the constitutive equations:

$$\operatorname{div} T = 0, \tag{11}$$

$$\operatorname{div} \mathbf{h} = 0, \tag{12}$$

$$R^{c^{-1}}{}_{ijkl} = 0, \tag{13}$$

where $R_{ijkl}{}^{C-l}$ is the Riemann tensor based on c^{-1} and T is symmetric. The constitutive equation of an incompressible material has the form $T = p\mathbf{1} + S$ (p being an arbitrary pressure) which yields

$$\operatorname{div} S = \operatorname{grad} p \tag{14}$$

instead of (11).

In the paper of Petroski and Carlson [1], the problem of finding all universal solutions for a homogeneous isotropic thermoelastic material is solved. It is shown that for $\mathbf{h}$ depending on θ the condition θ = constant results immediately. This is also true for Fourier material; hence we restrict our analysis to material of the form (8), (9) with

$$\varphi_i = \varphi_i(I, II). \tag{15}$$

4. Controllability conditions. Inserting T in (14) we get the same conditions as Petroski and Carlson [1]. Conditions (16)–(24) are the same conditions on which Ericksen's analysis [4] and results are based. The following tensors must be symmetric: (We will use direct notation where it is useful in the analysis.)

$$c^{-1i}{}_j|_{ik}\ , \tag{16}$$

$$c^i{}_j|_{ik}\ , \tag{17}$$

$$I|_k\,(c^{-1})^i{}_j|_i + [I|_i\,(c^{-1})^i{}_j]|_k\ , \tag{18}$$

$$II|_k\,(c^{-1})^i{}_j|_i + [II|_i\,(c^{-1})^i{}_j]|_k + I|_k\,c^i{}_j|_i - [I|_i\,c^i{}_j]|_k\ , \tag{19}$$

$$-II|_k\,c^i{}_j|_i - [II|_i\,c^i{}_j]|_k\ , \tag{20}$$

$$\nabla I \otimes c^{-1}\nabla I, \tag{21}$$

$$\nabla II \otimes c^{-1}\nabla I + \nabla I \otimes c^{-1}\nabla II - \nabla I \otimes c\nabla I, \tag{22}$$

$$\nabla II \otimes c^{-1}\nabla II - \nabla II \otimes c\nabla I - \nabla I \otimes c\nabla II, \tag{23}$$

$$-\nabla II \otimes c\nabla II, \tag{24}$$

$$[g_i(c^{-1})^i{}_j]|_k + g_k(c^{-1})^i{}_j|_i\ , \tag{25}$$

$$[g_i c^i{}_j]|_k + g_k c^i{}_j|_i\ , \tag{26}$$

$$\mathbf{g} \otimes c^{-1}\nabla I + \nabla I \otimes c^{-1}\mathbf{g}, \tag{27}$$

$$c^{-1}\mathbf{g} \otimes \mathbf{g}, \tag{28}$$

$$\mathbf{g} \otimes c^{-1}\nabla II - \mathbf{g} \otimes c\nabla I + \nabla II \otimes c^{-1}\mathbf{g} - \nabla I \otimes c\mathbf{g}, \tag{29}$$

$$\mathbf{g} \otimes c\nabla II + \nabla II \otimes c\mathbf{g}, \tag{30}$$

$$\mathbf{g} \otimes c\mathbf{g}. \tag{31}$$

In the case that $\mathbf{g} = \nabla\theta \neq 0$ holds, (28) and (31) yield that $\mathbf{g}$ is an eigenvector of c and c^{-1} and hence

$$c\mathbf{g} = \lambda\mathbf{g}, \tag{32}$$

$$c^{-1}\mathbf{g} = (1/\lambda)\mathbf{g}, \tag{31}$$

where $\lambda = \lambda(I, II)$.

Therefore we can draw two conclusions:

1) If c^{-1} and θ provide a universal solution, then the constitutive equation of h reduces to

$$\mathbf{h} = \left(\varphi_0 1 + \varphi_1\lambda 1 + \varphi_{-1}\frac{1}{\lambda}1\right)\mathbf{g} = \mu\mathbf{g} \tag{32}$$

where $\mu = \mu(I, II)$.

2) Universal solutions are not suitable for determining the constitutive equation of both thermoelastic and Fourier material. This is not possible except in the special case

of material for which

$$\mathbf{h} = \mu \mathbf{g} \tag{33}$$

Inserting (33) in (12) yields

$$\operatorname{div} \mu \mathbf{g} = \mu \operatorname{div} \mathbf{g} + \frac{\partial \mu}{\partial I} \nabla I \cdot \mathbf{g} + \frac{\partial \mu}{\partial II} \nabla II \cdot \mathbf{g} = 0$$

This equation is satisfied independently of the choice for μ if the following conditions hold:

$$\operatorname{div} \mathbf{g} = 0, \tag{34}$$

$$\mathbf{g} \cdot \nabla I = 0, \tag{35}$$

$$\mathbf{g} \cdot \nabla II = 0, \tag{36}$$

and from $\theta|^{i}{}_{;} = \theta|_{;}{}^{i}$ we get

$$g^{i}|_{i} = g_{i}|^{i}. \tag{37}$$

The conditions (34) and (37) do not yield the results which are obtained from Hamel's theorem, namely $\theta = k + p\varphi + qz$, as do the corresponding conditions in the analysis of Petroski and Carlson [1], e.g. $\theta = a \log r$ satisfies (34) and (37).

Many steps in the following analysis are analogous to the paper of Petroski and Carlson [1] but the proofs are made much simpler by directly applying the results obtained by different authors in the field.

We will consider two cases: A) I and II not both constant, B) I and II both constant.

5. *I* and *II* not both constant. From the paper of Ericksen (4) and Marris and Shiau [5] all c^{-1}'s solving the conditions (16)–(24) are known. These are the so-called families 1–4 (cf. [6]). In order to be universal solutions of our problem, these families have to satisfy (25)–(31) in addition. For these families Ericksen [4] has shown that if I and II are not both constant they are functionally dependent on a function β and the surfaces β = constant are parallel planes, concentric cylinders or concentric spheres. ∇I and ∇II are parallel to each other and in suitably chosen coordinate systems parallel to $\mathbf{e}_x$, $\mathbf{e}_r$ or $\mathbf{e}_\rho$, respectively. (We use (x, y, z) for cartesian coordinates, (r, φ, z) for cylindrical coordinates and (ρ, φ, ψ) for spherical coordinates; ψ is used here to avoid confusion with the temperature field θ.) If either I or II is constant, the above conclusions hold for the one that is not constant.

Conditions (35) and (36) yield that ∇I and ∇II are orthogonal to $\mathbf{g}$ unless I or II are constant. Next we show that ∇I, ∇II and $\mathbf{g}$ are eigenvectors to the same eigenvalue. Since they are eigenvectors of c and c^{-1}, we have:

$$c\mathbf{g} = \lambda \mathbf{g}, \qquad c^{-1}\mathbf{g} = \frac{1}{\lambda}\mathbf{g},$$

$$c\nabla I = A\nabla I, \qquad c^{-1}\nabla I = \frac{1}{A}\nabla I,$$

$$c\nabla II = A\nabla II, \qquad c^{-1}\nabla II = \frac{1}{A}\nabla II.$$

From (27) and (30) it follows that

$$\frac{1}{A}\nabla I \otimes g + \frac{1}{\lambda} g \otimes \nabla I \quad \text{(symmetric)}, \tag{38}$$

$$A\nabla II \otimes g + \lambda g \otimes \nabla II \quad \text{(symmetric)}. \tag{39}$$

Hence we get

$$\left(\frac{1}{A} - \frac{1}{\lambda}\right)(\nabla I \otimes g - g \otimes \nabla I) = 0 \tag{40}$$

$$(A - \lambda)(\nabla II \otimes g - g \otimes \nabla II) = 0. \tag{41}$$

If I is not constant, ∇I must be orthogonal to $\mathbf{g}$ and this yields $\nabla I \otimes g \neq g \otimes \nabla I$ and hence $1/A = 1/\lambda$. In case of $\nabla I = 0$, $\nabla II \neq 0$ holds and we have $A = \lambda$.

In order to find all universal solutions in this case we have to look for subfamilies of the families 1–4 with I and II not both constant for which the eigenvalues belonging to the eigenvectors $\mathbf{e}_x$, $\mathbf{e}_r$ or $\mathbf{e}_\rho$, respectively, are twofold. If there were such subfamilies the following equations would hold:

$$\operatorname{tr} c^{-1} = 2\lambda_1 + \lambda_3 \tag{42}$$

$$\det c^{-1} = \lambda_1^2\lambda_3 = 1 \tag{43}$$

Hence we have

$$\operatorname{tr} c^{-1} - 2\lambda_1 = 1/\lambda_1^2 \tag{44}$$

Eq. (44) is written out for each family in Table I, and it can easily be read off this table that there are no such subfamilies. We have used the same notation as Truesdell and Noll [6].

TABLE I.

Family 1	$\frac{A^2}{r^2} + B^2r^2 + B^2C^2 + \frac{1}{A^2B^2} - \frac{2A^2}{r^2} = \frac{r^4}{A^4}$
Family 2	$2AB^2x + \frac{1}{2Ax} + \frac{C^2}{2Ax} + \frac{1}{B^2} - 4AB^2x = \frac{1}{4A^2B^4x^2}$
Family 3	$+\frac{A}{r^2}(r^2 - B) + r^2\left(\frac{AC^2}{r^2 - B} + D^2\right) + \frac{AE^2}{r^2 - B} + F^2 - \frac{2A(r^2 - B)}{r^2} = \frac{r^4}{A^2(r^2 - B)^2}$ $\Leftrightarrow$ $A^2D^2r^8 + (A^3 + A^3C^2 - 2A^2BD^2 + A^2F - 2A^3 - 1)r^6 + (-3A^3B - A^3BC^2 + A^2B^2D^2 + A^3E^2 - 2A^2BF^2 - 6A^3B)r^4 + (3A^3B^2 - A^3BE^2 + A^2B^2F^2 + 6A^3B^2)r^2 + (-A^3B^2 + 2A^3B^3) = 0$ $\Rightarrow$ $D = B = E = \varnothing$ which gives a constant subfamily
Family 4	$R^{12} - 2r^6R^6 + r^{12} = 0$

6. I and II both constant. In this case only conditions (16), (17), (25), (26), (28) and (31) are not satisfied identically by c and c^{-1} and they are equivalent to the following equations:

$$c^i{}_j|_i = A g_j\,, \qquad \Rightarrow A = A(\theta), \quad \bar{A} = \bar{A}(\theta) \tag{45}$$

$$c^{-1i}{}_j|_i = \bar{A} g_j\,, \tag{46}$$

$$c\mathbf{g} = c_1 \mathbf{g}, \tag{47}$$

$$c^{-1}\mathbf{g} = \frac{1}{e_1}\mathbf{g}. \tag{48}$$

Setting $\mathbf{a}: = g/|g|$, we can write c and c^{-1} in the form:

$$c = (e_1 - e_3)\mathbf{a} \otimes \mathbf{a} + (e_2 - e_3)\mathbf{b} \otimes \mathbf{b} + e_3 \mathbf{1} \tag{49}$$

$$c^{-1} = \left(\frac{1}{e_1} - \frac{1}{e_3}\right)\mathbf{a} \otimes \mathbf{a} + \left(\frac{1}{e_2} - \frac{1}{e_3}\right)\mathbf{b} \otimes \mathbf{b} + \frac{1}{e_3}\,1 \tag{50}$$

where $\mathbf{a}$, $\mathbf{b}$ are unit vectors. Inserting this in (45) and (46) yields

$$\left.\left(\frac{\theta|^i\,\theta|_j}{\theta|^k\,\theta|_k}\right)\right|_i = F(\theta)\theta|_j \quad \text{if} \quad e_1 \neq e_3\,, \tag{51}$$

$$(b^i b_j)|_i = G(\theta)\theta|_j \quad \text{if} \quad e_2 \neq e_3\,. \tag{52}$$

Now there are two cases to be considered: a) all eigenvalues are different, b) two eigenvalues are equal. (The case $e_1 = e_2 = e_3$ is trivial).

a) In the case that all eigenvalues are different, (51) may be written in the form

$$a^i|_i a_j + a^i a_j|_i = F(\theta)(\theta|^k\theta|_k)^{1/2} a_j\,. \tag{53}$$

Since $|\mathbf{a}| = 1$ holds, we have $a_i|_j a^i a^j = 0$. Then (53) yields

$$a^i|_i = F(\theta), \tag{54}$$

$$a^i a_j|_i = 0. \tag{55}$$

Using the same arguments as Ericksen [4], we can conclude from (55) that $\nabla(\nabla\theta^2)$ and $\nabla\theta$ are parallel, so that $|\nabla\theta| = f(\theta)$ holds, and for the Gaussian curvature K_1 and the total curvature K_2 of the surfaces $\theta =$ constant we find

$$K_1 = a^i|_i = K_1(\theta), \tag{56}$$

$$K_2 = \tfrac{1}{2}[(a^i|_i)^2 - a^i|^j\, a_j|_i - a^i|_j\, a^j a_i|_k\, a^k] = K_2(\theta). \tag{57}$$

Hence the surfaces $\theta =$ constant are planes, circular cylinders or spherical shells.

Considering curves $\mathbf{x} = \mathbf{x}(s)$ with $d\mathbf{x}/ds = \mathbf{a}$, we find

$$d^2\mathbf{x}/ds^2 = \frac{d}{ds}\mathbf{a} = \mathbf{a} \otimes \nabla\frac{d\mathbf{x}}{ds} = a_i|_j\, a^j = 0 \tag{58}$$

These curves are straight lines and therefore the surfaces $\theta =$ constant are parallel planes, concentric circular cylinders or concentric spherical shells.

a1) The surfaces $\theta =$ constant are parallel planes. In this case there is a cartesian coordinate system (x, y, z) for which $\theta = \theta(x)$ holds. Since $\mathbf{b} \perp \mathbf{g}$ we have $b_1 = 0$, $b_2 = \cos \eta$, $b_3 = \sin \eta$ and from (52)

$$b^i|_i = 0 \quad \text{and} \quad b^i b_j|_i = (b_j|_i - b_i|_j)b^i = 0 \tag{59}$$

Hence

$$\begin{aligned} -\sin \eta \eta_y + \cos \eta \eta_z &= 0 \\ \cos \eta \eta_y + \sin \eta \eta_z &= 0 \end{aligned} \Rightarrow \eta_y = \eta_z = 0$$

and c^{-1} depends on j only.

From the condition $R_{1212}{}^{c^{-1}} = 0$ we get $\eta_x{}^2 = 0$ and hence $\eta =$ constant. In this case the homogeneous deformation and the temperature field $\theta = ax + b$ are the only possible solution which satisfies (34) with $\theta = \theta(x)$.

a2) The surfaces $\theta =$ constant are concentric circular cylinders. In this case there is a cylindrical coordinate system (r, φ, z) such that $\theta = \theta(r)$ holds. In the above manner we can conclude that $b_1 = 0$, $b_2 = r \cos \eta$, $b_3 = r \sin \eta$ and $b^i|_i = 0$, $b^i b_j|_i = (b_j|_i - b_i|_j)b^i = 0$ holds. Hence

$$\begin{aligned} -r \sin \eta \eta_\varphi + \cos \eta \eta_z &= 0 \\ \cos \eta \eta_\varphi + r \sin \eta \eta_z &= 0 \end{aligned} \Rightarrow \eta_\varphi = \eta_z = 0$$

and c^{-1} depends on r only. Here Ericksen's analysis yields that the eigenvalues cannot be constant unless η is constant; this leads to

$$c^{-1} = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$$

and $\theta = \theta(r)$ must satisfy the Eq. (34):

$$\theta_{rr} + \frac{1}{r}\,\theta_r = 0 \tag{60}$$

for which $\theta = \mathrm{a} \log r + b$ provides the general solution.

a3) The surfaces $\theta =$ constant are concentric spherical shells. In this case there exists a spherical coordinate system (ρ, φ, ψ) such that $\theta = \theta(\rho)$, and in the above manner we have $b_1 = 0$, $b_2 = \rho \cos \eta$, $b_3 = \rho \sin \varphi \sin \eta$ and $b^i|_i = 0$, $b^i b_j|_i = 0$. Hence

$$\frac{1}{\rho \sin \varphi}\left(\frac{\partial}{\partial \varphi} \sin \varphi \cos \eta + \frac{\partial}{\partial \psi} \sin \eta\right) = 0$$

$$\rho\left(\frac{\partial}{\partial \psi} \cos \eta - \frac{\partial}{\partial \varphi} \sin \varphi \sin \eta\right) = 0$$

$\Rightarrow \eta_\varphi = 0$, $\eta_\psi = -\cos \varphi$, $\Rightarrow \eta =$ constant,
and by the same argument that Ericksen uses, we have

$$c^{-1} = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}.$$

$\theta = \theta(\rho)$ must satisfy $\theta_{\rho\rho} + 2/\rho(\theta_\rho) = 0$ and therefore it must be of the form $\theta = a\rho^{-1}+b$.

b) In the case that two eigenvalues are equal, c can be written in the form

$$c = (e_1 - e_3)\mathbf{b} \otimes \mathbf{b} + e_3\mathbf{1}. \tag{61}$$

In addition, $\nabla I = \nabla II = 0$ holds. As Kafadar [7] has shown, these conditions yield $\mathbf{b}$ = constant. Hence we can find a cartesian coordinate system (x, y, z) for which $\mathbf{b} = \mathbf{e}_x$ holds and $\mathbf{b}$ can be written in the form (1, 0, 0) and c^{-1} must be of the type

$$c^{-1} = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_2 \end{bmatrix}.$$

Now there are two possibilities:

b1) $\mathbf{b} \parallel \mathbf{g}$. Here we have $\mathbf{g}/|\mathbf{g}|$ = constant and the surfaces θ = constant are the planes x = constant and from this and (34) we obtain $\theta = ax + b$.

b) $\mathbf{b} \perp \mathbf{g}$. This gives: $\mathbf{b}\cdot\mathbf{g} = \theta_x = 0$. Hence θ does not depend on x and it must be of the form $\theta = \theta(y, z)$ and has to satisfy (34) which reads in this case

$$\Delta_2\theta = \theta_{yy} + \theta_{zz} = 0. \tag{62}$$

Therefore any plane rotation-free temperature field provides a solution.

7. Universal solutions for compressible material. The Cauchy stress tensor T of a compressible homogeneous isotropic thermoelastic material may be written in the form:

$$T = 2\rho \sum_{n=1}^{3} \frac{\partial\psi}{\partial I_n} \frac{\partial I_n}{\partial c^{-1}} c^{-1} \tag{63}$$

where $\psi = \psi(\theta, I, II, III)$ and $III = \det c^{-1}$. Inserting (63) in (11) leads to

$$\operatorname{div} T = 2\rho \sum_{n=1}^{3} \sum_{m=1}^{3} \frac{\partial^2\psi}{\partial I_n\,\partial I_m} \nabla I_m \frac{\partial I_n}{\partial c^{-1}} c^{-1} + 2\sum_{n=1}^{3} \frac{\partial\psi}{\partial I_n} \nabla\left[\rho \frac{\partial I_n}{\partial c^{-1}} c^{-1}\right] + 2\rho \sum_{n=1}^{3} \frac{\partial^2\psi}{\partial I_n\,\partial\theta} \nabla\theta \frac{\partial I_n}{\partial c^{-1}} c^{-1} = 0. \tag{64}$$

The first two right-hand-side expressions yield the conditions used by Ericksen [8] to show that the homogeneous deformation is the only solution. For $n = 3$ the last term gives

$$\nabla\theta = 0 \Rightarrow \theta = \text{constant}. \tag{65}$$

To obtain a non-trivial solution for the temperature field we must restrict our analysis to materials with

$$\psi = \psi(I, II, III). \tag{66}$$

In this case Eq. (64) is satisfied by the homogeneous deformation and gives no restriction for the temperature field.

The constitutive equation for the heat flux $\mathbf{h}$ does not reduce for compressible Fourier material and we have to evaluate Eq. (12) with $\mathbf{h}$ of the form (g) and φ_i =

$\varphi_i(I, II, III)$. Since the invariants are constant, the coefficient functions are constant, too, and Eq. (12) yields:

$$\operatorname{div} \mathbf{g} = 0, \tag{67}$$

$$\operatorname{div} c\mathbf{g} = 0, \tag{68}$$

$$\operatorname{div} c^{-1}\mathbf{g} = 0, \tag{69}$$

where we have made use of the fact that c and c^{-1} are constant.

Now we introduce the notation $G^i{}_i : \theta|^i{}_i$ and rewrite (67)–(69) in the form:

$$\operatorname{tr} G = 0, \tag{70}$$

$$\operatorname{tr} cG = 0, \tag{71}$$

$$\operatorname{tr} c^{-1}G = 0, \tag{72}$$

For every homogeneous deformation there is a cartesian coordinate system (x, y, z) such that c may be written in the form

$$c = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix} \tag{73}$$

and Eqs. (70)–(72) are equivalent to

$$\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^{-2} & \lambda_2^{-2} & \lambda_3^{-2} \end{pmatrix} \begin{pmatrix} \theta_{xx} \\ \theta_{yy} \\ \theta_{zz} \end{pmatrix} = 0. \tag{74}$$

There are three possibilities:

a) All eigenvalues are different. Then the only solution of (74) is $\theta_{xx} = \theta_{yy} = \theta_{zz} = 0$ and θ must be a linear function of x, y and z.

b) Two eigenvalues are equal, say $\lambda_1^2 = \lambda_2^2$. Then the only solution of (74) is $\theta_{xx} + \theta_{yy} = 0$, that is, θ must be an arbitrary plane rotation-free field orthogonal to the main stretch.

c) All eigenvalues are equal. Then (74) yields $\Delta\theta = \theta_{xx} + \theta_{yy} + \theta_{zz} = 0$. Therefore θ may be an arbitrary rotation-free field.

8. Results. For homogeneous isotropic incompressible Fourier material we have found the following four types of universal solutions:

1) Stretching and shearing of a rectangular block together with a constant gradient temperature field in the direction of the stretch:

$$x = AX,\ y = BY + CZ,\ z = DY + EZ,\ \theta = ax + b,\ A(BE - CD) = 1.$$

2) Inflating and stretching of a sector of an annular wedge together with an axial temperature field:

$$r = AR,\ \varphi = B\phi,\ z = (A^2B)^{-1}Z,\ \theta = a \log r + b.$$

3) Inflating and azimuthal contracting of a sector of a spherical shell together with a radial temperature field.

$$\rho = AP,\ \varphi = \pm\phi,\ \psi = \mp A\Psi,\ \theta = a\rho^{-1} + b,$$

4) Stretching of a rectangular block with equal transverse contractions together with an arbitrary plane rotation-free temperature field orthogonal to the stretch.

$$x = A^2X,\ y = A^{-1}Y,\ z = A^{-1}Z,$$

θ may be any solution of $\theta_{yy} + \theta_{zz} = 0$.(A, B, C, D and E and a, b are constants.) Petroski and Carlson [1] found that only type 1 and type 4 with $\theta = a\varphi$ are universal solutions for homogeneous incompressible isotropic thermoelastic material.

For compressible material we had to restrict ourselves to Fourier material with T depending on c only and not on θ. We have determined the following universal solutions:

a) homogeneous deformation with any temperature field obeying $\theta_{xx} = \theta_{yy} = \theta_{yy} = 0$.

b) homogeneous deformation with two equal eigenvalues and any temperature field obeying $\theta_{xx} + \theta_{yy} = 0$ ($\mathbf{e}_x$ and $\mathbf{e}_y$ being the corresponding eigenvectors).

c) homogeneous inflation or contraction with any temperature field obeying $\Delta\theta = \theta_{xx} + \theta_{yy} + \theta_{zz} = 0$.

The above analysis shows that the Fourier subclass of thermoelastic material differs from the general thermoelastic material by its universal solutions which provide the basis for experimental tests to find out whether or not a given material is Fourier, in the sense that failure proves that it is not Fourier. Thus the mathematical theory justifies this classification of thermoelastic material which would make no sense if no ways of experimentally distinguishing between the two classes were available.

On the other hand, the universal solutions found in our analysis are no help in determining the constitutive equations of the heat flux of an incompressible material. The response of Fourier material to universal solutions is determined by only one function while its general response requires three functions.

We conclude with two observations. The existence of universal solutions for Fourier material suggests further investigation of this subclass of thermoelastic materials. But the impossibility of determining the constitutive equations raises some doubts about the concept of universal solution and its usefulness. A class of materials defined by constitutive equations, that do not admit any universal solutions with the property that the response functions can be uniquely determined from them, makes no sense to the experimentalist.

References

[1] Petroski and Carlson, Arch. Rat. Mech. Anal. **31,** 127–150 (1968)
[2] Hamel, Sitz.-Ber. Preuss. Akad. Wiss. **5,** 5–20 (1937)
[3] Truesdell, *Rational thermodynamics*, New York (1969)
[4] Ericksen, ZAMP **5** (1954), 466–486
[5] Marris and Shiau, Arch. Rat. Mech. Anal. **36/2** (1970), 135–160
[6] Truesdell and Noll, Handbuch der Physik III/3, Sec. 57
[7] Kafadar, Arch. Rat. Mech. Anal. **47,** 15–27 (1972)
[8] Ericksen, J. Math. Phys. **34,** 126–128 (1955)
[9] Hayes, Laws & Osborn, Quart. Appl. Math. **27,** 416–420 (1969)