NOTE ON A PAPER BY ANDERSON AND ARTHURS*

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It is interesting to examine in terms of the hypercircle the problem discussed by Anderson and Arthurs. But it is illuminating to generalize the problem a little. Generalization to n dimensions is trivial, so let us think of a plane with rectangular cartesian coordinates x_i. Subscripts take the values 1, 2, and summation is understood for repeated suffixes. Partial differentiation is indicated by a comma, so that $\phi_{,i} = \partial\phi/\partial x_i$.

Consider the partial differential equation

$$\phi_{,ii} + f_i\phi_{,i} + C = 0, \tag{1}$$

to be solved in a domain V with the boundary condition $\phi = 0$. Here C is any given function of position and so are f_i, subject to the condition that $f_i\, dx_i$ is an exact differential, so that, if we define

$$F = \exp \int f_i\, dx_i\ , \tag{2}$$

we have

$$F_{,i}/F = f_i\ . \tag{3}$$

Thus (1) may be written

$$(F\phi_{,i})_{,i} + D = 0, \qquad D = FC. \tag{4}$$

Consider a Hilbert space in which the point corresponds to a vector field p_i in V, with the inner product

$$\int p_i p_i' F\, dV \qquad (dV = dx_1\, dx_2). \tag{5}$$

Consider two linear subspaces defined as follows: L' : p_1' where $(Fp_i')_{,i} + D = 0$; L'' : p_i'' where $p_i'' = v_{,i}$, the scalar v being arbitrary except that $v = 0$ on the boundary. It is clear that the solution of the problem is the intersection of L' and L''. Moreover L' is orthogonal to L''. To see this, take any Hilbertian vector *lying in* L', that is, the join of any two points in L'; denoting it by q_i', we have

$$(Fq_i')_{,i} = 0. \tag{6}$$

Then the inner product of a vector lying in L' with one lying in L'' is

$$\int q_i'p_i''F\, dV = \int v_{,i}Fq_i'\, dV = 0, \tag{7}$$

by (6) and the boundary condition $v = 0$.

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Thus the gradient $\phi_{,i}$ of the solution of (1) appears in Hilbert space as the unique intersection of two mutually orthogonal linear subspaces L', L'', and the method of the hypercircle is immediately applicable. But to get a good approximation to $\phi_{,i}$ (in the mean-square sense) we need two sequences of Hilbertian points, one in L' and the other in L'', converging on the unique intersection. L'' is relatively easy to deal with, because we need only functions v which vanish on the boundary, corresponding to the trial functions in Eq. (16) of Anderson and Arthurs. Their trial functions (17) are scalars satisfying their partial differential equation (10) without boundary conditions. In the hypercircle approach we are a little freer, because to get Hilbertian points in L' we need only satisfy

$$(Fp_i')_{,i} + D = 0, \tag{8}$$

again without boundary conditions. The best procedure here will depend on the form of the functions F and D, but we might at least recommend finding some particular solution of this equation (perhaps with $p_2' = 0$), thus getting one Hilbertian point on L', and then spreading out over L' by means of vectors lying in L'; these are to satisfy (6), and so we might choose an arbitrary vector field w_i and write

$$Fq_i' = \epsilon_{ijk} w_{j,k} , \tag{9}$$

using the permutation symbol (here it is pertinent that we are in a plane).

It is not suggested that the method of the hypercircle is superior computationally to that of Anderson and Arthurs. It provides a different and more geometrical approach. For an elaborate treatment of the hypercircle method, see [1], and for a brief account, [2].

References

[1] J. L. Synge, *The hypercircle in mathematical physics*, Cambridge University Press, 1957

[2] J. L. Synge, *The hypercircle method*, in *Studies in numerical analysis*, Academic Press Inc. (London) Ltd., 1974, pp. 201–217