COMPLEMENTARY VARIATIONAL PRINCIPLES FOR A BOUNDARY-VALUE PROBLEM IN TWISTING OF RING SECTORS*

BY N. ANDERSON AND A. M. ARTHURS (*University of York, England*)

Abstract. Complementary variational principles associated with the boundary-value problem for stresses in close-coiled helical springs are presented. An accurate variational solution is obtained for an illustrative problem.

The boundary-value problem

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{3}{R - x}\frac{\partial\phi}{\partial x} + 2 = 0 \quad \text{in} \quad V, \tag{1}$$

$$\phi = 0 \quad \text{on} \quad \partial V, \tag{2}$$

arises in the theory of stresses in close-coiled helical springs [1]. Here ϕ is the stress function, V is the circular region $x^2 + y^2 \leq a^2$, a being the radius of a cross-section of the ring, ∂V is the boundary of V, and $R(>a)$ is the radius of the helical coil. Timoshenko [1] showed how to solve this problem by successive approximations using the fact that $a/R < 1$. He obtained

$$\phi \simeq \phi_T = \phi_0 + \phi_1 + \phi_2\,, \tag{3}$$

where

$$\phi_0 = -\tfrac{1}{2}(x^2 + y^2 - a^2), \tag{4}$$

$$\phi_1 = \frac{3}{8R}\, x(x^2 + y^2 - a^2), \tag{5}$$

$$\phi_2 = -\frac{1}{64R^2}\,(x^2 + 5y^2 - 15a^2)(x^2 + y^2 - a^2). \tag{6}$$

Variational methods provide an alternative means of solving certain boundary-value problems, and in this paper we show how the theory of complementary variational principles [2] can be used to find an accurate variational solution of (1) and (2).

To obtain variational principles associated with boundary value problem in (1) and (2), we rewrite (1) in canonical form:

$$T\phi = u = \partial H/\partial u, \qquad T^*u = 2 = \partial H/\partial\phi,$$

where $T = \text{grad}$, $T^* = -(R - x)^3 \operatorname{div}\{(R - x)^{-3}\}$, T^* being the formal adjoint of T with respect to the inner product $\langle\phi, \psi\rangle = \int_V \phi\psi(R - x)^{-3}\,dxdy$. Following the general procedure for canonical equations [2], we are led to the functionals

$$J(\Phi) = \int_V \{\tfrac{1}{2}(\text{grad } \Phi)^2 - 2\Phi\}(R - x)^{-3}\,dxdy, \tag{7}$$

* Received December 14, 1973.

with

$$\Phi = 0 \quad \text{on} \quad \partial V, \tag{8}$$

and

$$G(\Psi) = -\frac{1}{2}\int_V (\text{grad }\Psi)^2 (R - x)^{-3}\, dxdy, \tag{9}$$

with

$$L\Psi \equiv -(R - x)^3 \,\text{div}\,\{(R - x)^{-3}\,\text{grad }\Psi\} = 2 \text{ in } V. \tag{10}$$

Then it can readily be shown that these functionals are stationary at the exact solution ϕ of the boundary-value problem in (1) and (2). In addition, the global complementary variational principles

$$G(\Psi) \leq G(\phi) = J(\phi) \leq J(\Phi) \tag{11}$$

hold, equality arising when Φ and Ψ are equal to ϕ.

For approximate variational solutions Φ an estimate of the error is available from the error bound [3]

$$\|\Phi - \phi\|_{L^2} \leq E(\Phi), \tag{12}$$

where $\|\psi\|_{L^2}$ is the L^2 norm $\{\int_V \psi^2 (R - x)^{-3}\, dxdy\}^{1/2}$, and

$$E(\Phi) = [2\Lambda^{-1}\{J(\Phi) - G(\Psi)\}]^{1/2}. \tag{13}$$

Here Λ is a lower bound to the lowest eigenvalue of the problem

$$L\theta = \lambda\theta \quad \text{in} \quad V, \tag{14}$$

$$\theta = 0 \quad \text{on} \quad \partial V, \tag{15}$$

the operator L being defined by Eq. (10).

We have performed calculations of the complementary functionals in the case $a = 1$, $R = 10$. As trial functions we took

$$\Phi = \sum_{n=1}^{6} (a_n x + b_n)(x^2 + y^2 - a^2)^n, \tag{16}$$

which satisfies the condition (8), and

$$\Psi = c_1(R - x)^2 + c_2(R - x)^4 + (2c_1 - 1)y^2, \tag{17}$$

which satisfies the constraint (10). The parameters a_n, b_n, c_1 and c_2 were found by optimizing J and G. To estimate the error in Φ from (13) we require Λ, which for this case can be taken as

$$\Lambda = 6, \tag{18}$$

and the results including an error bound are given in Table 1. From these results we see that the variational solution Φ is quite accurate. Using (3) and (16) we find that for $a = 1$, $R = 10$,

$$\phi_T(0, 0) = 0.4977, \qquad \Phi(0, 0) = 0.4984, \tag{19}$$

TABLE 1

Variational parameters and error bound for $a = 1$, $R = 10$. Here $m(-n)$ means $m \times 10^{-n}$.

a_1	0.281250(−1)	b_1	−0.485001	c_1	0.250959
a_2	−0.618750(−1)	b_2	0.771875(−1)	c_2	−0.124541(−2)
a_3	−0.122812	b_3	0.137500	J	−0.7868(−3)
a_4	−0.828125(−1)	b_4	0.821875(−1)	G	−0.7876(−3)
a_5	−0.937500(−2)	b_5	0.687500(−2)	E	0.0005
a_6	0.156250(−2)	b_6	−0.156250(−2)		

indicating good agreement between the perturbation solution (3) and the variational solution (16).

One advantage of the variational approach is that it is not restricted, as the perturbation solution (3) is, to very small values of the parameter a/R.

For an alternate analytical solution of this problem see the paper by Freiberger [4].

References

[1] S. Timoshenko, *Theory of elasticity*, New York, 1934, pp. 355–361
[2] A. M. Arthurs, *Complementary variational principles*, Oxford, 1970
[3] A. M. Arthurs, J. Math. Anal. Appl. **41**, 781–795 (1973)
[4] W. Freiberger, Australian J. Sci. Res. **2**, 354–375 (1949)