INFINITELY STRETCHED MOONEY SURFACES OF REVOLUTION ARE UNIFORMLY STRESSED CATENOIDS*

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Abstract. Axially and radially stretched Mooney surfaces of revolution are found to tend to catenoids as the stretching tends to infinity. Moreover, two catenoids are found to exist for any given set of stretching parameters. A formal two-term asymptotic solution is obtained explicitly and the stretching of a cylindrical surface is given as an example.

1. Introduction. Isaacson [1] showed in 1965 that the shape of an inflated arbitrary Mooney surface of revolution tends to a spherical surface as the inflating pressure tends to infinity. The radius of the spherical surface as well as a formal two-term asymptotic solution have recently been obtained by Wu [2]. These results follow from the fact that an infinitely stretched Mooney surface is uniformly stressed and the fact that the equilibrium configuration of a uniformly stressed closed surface subjected to a constant normal pressure is spherical.

It is also known that the equilibrium configuration of a uniformly stressed surface subjected to no surface load is a minimal surface, of which a flat surface is a special case. The flattening of membranes of revolution by large stretching is of this nature and the relevant result was obtained by Perng and Wu [3]. The most general minimal surface of revolution is the catenoid [4]. In fact one of the very few nonlinear membrane problems that can be solved explicitly deals with the stretching of catenoids [5, p. 162].

The above observations suggest that the equilibrium configuration of an arbitrary Mooney surface of revolution subjected to radial and axial stretch tends to a catenoid as the stretching becomes infinitely large. A formal asymptotic analysis presented in this paper shows that this is indeed the case. Moreover, two asymptotic states exist for a given set of boundary conditions. The two states, however, are associated with different potential energy levels. If we assume that nature prefers the state of a lower energy, then the state with a lower energy may be called the preferred state.

2. Formulation. Let (r, θ, z) be a fixed cylindrical coordinate system and let S measure the dimensionless arc length along a curve C defined by

$$C: \begin{matrix} r = R(S) \\ z = Z(S) \end{matrix}, \qquad R(S) > 0, \quad 0 \le S \le 1, \tag{2.1}$$

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characterizing the meridian curve of a membrane of revolution. The functions R and Z are continuous but may have discontinuous derivatives. The membrane is stretched axisymetrically so that the deformed shape can be characterized by a meridian curve

$$c: r = X(S), \qquad z = Y(S), \tag{2.2}$$

satisfying the conditions

$$X(0) = a\epsilon^{-1/2}, \qquad X(1) = b\epsilon^{-1/2}, \tag{2.3}$$

$$Y(1) - Y(0) = 2\epsilon^{-1/2}, \tag{2.4}$$

where $\epsilon > 0$ is a small parameter. The condition (2.4) determines the curve c to within a rigid-body displacement along the z-axis. The constants a and b are positive. We assume without loss of generality that $b \geq a$. (We can always redefine $R(S)$ to satisfy this condition.) The deformation from (2.1) to (2.2) is the subject of our investigation.

Suppose we denote by Λ_1 and Λ_2, respectively, the principal extension ratios in the meridian and azimuthal directions; then

$$\Lambda_1 = dL/dS, \qquad \Lambda_2 = X/R, \tag{2.5, 2.6}$$

where $L = L(S)$ measures the arc length along the curve c. We assume that the membrane is made of a Mooney material [5] characterized by a strain-energy function W defined by

$$W(\Lambda_1, \Lambda_2) = \left(\Lambda_1^2 + \Lambda_2^2 + \frac{1}{\Lambda_1^2\Lambda_1^2}\right) + k\left(\Lambda_1^2\Lambda_2^2 + \frac{1}{\Lambda_1^2} + \frac{1}{\Lambda_2^2}\right) \tag{2.7}$$

where $k = C_2/C_1$, the ratio of the two Mooney constants, and W is nondimensionalized by the quantity C_1H, H being the constant thickness of the undeformed membrane.

Based on (2.7), the fundamental equations can be derived. We prefer to use the set of equations given in [6]. These are

$$T_1 = \frac{1}{\Lambda_2} W_{\Lambda_1}, \qquad T_2 = \frac{1}{\Lambda_1} W_{\Lambda_2}, \tag{2.8, 9}$$

$$X\frac{dT_1}{dS} = (T_2 - T_1)\frac{dX}{dS}, \qquad \frac{d}{dS}(XT_1 \sin\Phi) = 0, \tag{2.10, 11)*}$$

$$\frac{1}{\Lambda_1}\frac{dX}{dS} = \cos\Phi, \qquad \frac{1}{\Lambda_1}\frac{dY}{dS} = \sin\Phi, \tag{2.12, 13}$$

where the subscripts on W denote partial differentiation with respect to the indicated argument. T_1 and T_2 are, respectively, the meridian and azimuthal stress resultants. The function Φ is the angle between the tangent to c and the r-axis. Eqs. (2.5)–(2.13), together with the conditions (2.3) and (2.4), constitute the complete formulation of the problem. We repeat, however, that the function $Y(S)$ can be determined only to within an arbitrary constant. Moreover, the origin of $L(S)$ is not specified. The arbitrariness does not alter the nature of the physical problem but enables us to cast our final result into a more convenient form.

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* (2.11) is the first integral of the original equation and represents the equilibrium condition along the z-axis.

Our objective now is to solve the posed problem asymptotically in terms of the parameter ϵ as ϵ tends to zero. Our analysis will be formal and the order symbol will refer to the parameter ϵ as $\epsilon \to 0$.

The conditions (2.3) and (2.4) suggest that the functions X and Y are $O(\epsilon^{-1/2})$. The orders of the other quantities follow accordingly. We write

$$\begin{aligned} X &= \epsilon^{-1/2}x, \quad Y = \epsilon^{-1/2}y, \quad L = \epsilon^{-1/2}l, \quad \Lambda_1 = \epsilon^{-1/2}\lambda_1 , \\ \Lambda_2 &= \epsilon^{-1/2}\lambda_2 , \quad T_1 = \epsilon^{-1}t_1 , \quad T_2 = \epsilon^{-1}t_2 , \quad \Phi = \phi. \end{aligned} \tag{2.14}$$

Eq. (2.7) can now be written as

$$W(\Lambda_1 , \Lambda_2) = \epsilon^{-2}w(\lambda_1 , \lambda_2 , \epsilon) \tag{2.15}$$

where

$$w(\lambda_1 , \lambda_2 , \epsilon) = k\lambda_2^2\lambda_1^2 + \epsilon(\lambda_1^2 + \lambda_2^2) + \epsilon^3 k\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right) + \epsilon^4 \frac{1}{\lambda_1^2\lambda_2^2}. \tag{2.16}$$

Because of (2.16), we shall consider all the newly introduced quantities in (2.14) as functions of S and ϵ and write $f = f(S, \epsilon)$ where f is a generic symbol. We shall assume that f is analytic in S and ϵ, and adopt the convenient notation:

$$\frac{d^n f}{dS^n} \equiv \frac{\partial^n f}{\partial S^n} , \qquad f_n(S) \equiv \frac{\partial^n}{\partial \epsilon^n} f(S, \epsilon) \bigg|_{\epsilon=0} . \tag{2.17}$$

The governing equations (2.5), (2.6) and (2.8–2.13), and the boundary conditions (2.3) and (2.4) now become

$$\lambda_1 = dl/dS, \qquad \lambda_2 = x/R, \tag{2.18, 19}$$

$$t_1 = \frac{1}{\lambda_2} w_{\lambda_1} , \qquad t_2 = \frac{1}{\lambda_1} w_{\lambda_2} , \tag{2.20, 21}$$

$$x \frac{dt_1}{dS} = (t_2 - t_1) \frac{dx}{dS} , \qquad \frac{d}{dS} (xt_1 \sin \phi) = 0, \tag{2.22, 23}$$

$$\frac{1}{\lambda_1} \frac{dx}{dS} = \cos \phi, \qquad \frac{1}{\lambda_1} \frac{dy}{dS} = \sin \phi, \tag{2.24, 25}$$

$$x(0, \epsilon) = a, \quad x(1, \epsilon) = b, \quad y(1, \epsilon) - y(0, \epsilon) = 2. \tag{2.26}$$

3. Asymptotic configuration—catenoid. The asymptotic state is an infinitely stretched state. If we let $\epsilon = 0$ and use the notation (2.17), the system of equations (2.18)–(2.26) becomes

$$\lambda_{10} = dl_0/dS, \qquad \lambda_{20} = x_0/R, \tag{3.1, 2}$$

$$t_{10} = 2k\lambda_{10}\lambda_{20} , \qquad t_{20} = 2k\lambda_{10}\lambda_{20} , \tag{3.3, 4}$$

$$x_0 \frac{dt_{10}}{dS} = (t_{20} - t_{10}) \frac{dx_0}{dS} , \qquad \frac{d}{dS} (x_0 t_{10} \sin \phi_0) = 0, \tag{3.5, 6}$$

$$\frac{1}{\lambda_{10}} \frac{dx_0}{dS} = \cos \phi_0 , \qquad \frac{1}{\lambda_{10}} \frac{dy_0}{dS} = \sin \phi_0 , \tag{3.7, 8}$$

$$x_0(0) = a, \quad x_0(1) = b, \quad y_0(1) - y_0(0) = 2. \tag{3.9}$$

It follows from (3.3), (3.4) and (3.5) that

$$t_{10} = t_{20} = \text{constant}. \tag{3.10}$$

Eq. (3.6) yields

$$x_0 = \frac{f_0}{t_{10}} \frac{1}{\sin \phi_0} \tag{3.11}$$

where f_0 is as unknown integration constant. Eq. (3.6), together with (3.1), (3.7) and (3.11), implies

$$dl_0 = -\frac{f_0}{t_{10}} \frac{1}{\sin^2 \phi_0} d\phi_0 . \tag{3.12}$$

Eq. (3.8), together with (3.1) and (3.12), implies

$$dy_0 = -\frac{f_0}{t_{10}} \frac{1}{\sin \phi_0} d\phi_0 \tag{3.13}$$

which, in turn, yields

$$y_0 = -\frac{f_0}{t_{10}} \ln \tan \frac{\phi_0}{2}. \tag{3.14}$$

Note that the boundary condition is not violated by our setting the integration constant to zero in (3.14). Eqs. (3.11) and (3.14) are the coordinates of a catenary which can also be written as

$$x_0 = \frac{f_0}{t_{10}} \cosh \frac{t_{10}}{f_0} y_0 . \tag{3.15}$$

Substituting (3.1) and (3.2) into (3.3) and applying (3.12), we get

$$\frac{t_{10}}{2k} R(S)\, dS = -\left(\frac{f_0}{t_{10}} \frac{1}{\sin \phi_0}\right)^2 \frac{d\phi_0}{\sin \phi_0}$$

or, after applying (3.11), (3.13) and (3.15),

$$\frac{dy_0}{dS} = \frac{t_{10}}{2k} \frac{t_{10}}{f_0} \frac{R(S)}{\cosh^2 \frac{t_{10}}{f_0} y_0}. \tag{3.16}$$

Integrating (3.16) yields

$$\frac{t_{10}}{k}\left(\frac{t_{10}}{f_0}\right)^2 \int_0^S R(S')\, dS'$$
$$= \frac{1}{2}\left\{\sinh\left[2\frac{t_{10}}{f_0} y_0(S)\right] - \sinh\left[2\frac{t_{10}}{f_0} y_0(0)\right]\right\} + \frac{t_{10}}{f_0}[y_0(S) - y_0(0)] \tag{3.17}$$

where $y_0(0)$ is an unknown integration constant. Since $y_0(S)$ must be consistant with our previous choice (3.14), $y_0(0)$ cannot be arbitrary. The function $y_0(S)$ is given implicitly by (3.17). All the other quantities can be determined by simple substitution. In particular, (3.12) yields

$$l_0 = \frac{f_0}{t_{10}} \sinh \frac{t_{10}}{f_0} y_0 \tag{3.18}$$

where the origin of l_0 is chosen at the point $y_0 = 0$.

We must now determine the three constants t_{10}, f_0 and $u_0(0)$. Eq. (3.9) requires that

$$a = \frac{f_0}{t_{10}} \cosh \frac{t_{10}}{f_0} y_0(0), \tag{3.19}$$

$$b = \frac{f_0}{t_{10}} \cosh \frac{t_{10}}{f_0} (y_0(0) + 2), \tag{3.20}$$

$$\frac{t_{10}}{k} \left(\frac{t_{10}}{f_0}\right)^2 R_c = 2 \frac{t_{10}}{f_0} + \frac{1}{2} \left[\sinh 2 \frac{t_{10}}{f_0} (y_0(0) + 2) - \sinh 2 \frac{t_{10}}{f_0} y_0(0)\right] \tag{3.21}$$

where

$$R_c \equiv \int_0^1 R(S)\, dS. \tag{3.22}$$

Eqs. (3.9), (3.15) and the assumption $b \geq a$ imply that $y_0(1) = y_0(0) + 2 \geq 1$. It follows from (3.19) and (3.20) that

$$\frac{t_{10}}{f_0} y_0(0) = \mp \cosh^{-1} \frac{t_{10}}{f_0} a, \tag{3.23}$$

$$\frac{t_{10}}{f_0} (y_0(0) + 2) = +\cosh^{-1} \frac{t_{10}}{f_0} b, \tag{3.24}$$

which, in turn, yield

$$\frac{t_{10}}{f_0} = \frac{1}{2} \left[\cosh^{-1} \frac{t_{10}}{f_0} b \pm \cosh^{-1} \frac{t_{10}}{f_0} a\right]. \tag{3.25}$$

For given a and b, (3.25) determines the ratio t_{10}/f_0 which can then be used to obtain $y_0(0)$ from (3.23). We note in passing that (3.25) has two roots in general.

Subtracting (3.19) from (3.20) and simplifying yields

$$\sinh \frac{t_{10}}{f_0} (y_0(0) + 1) = \frac{t_{10}}{f_0} \frac{b - a}{2 \sinh \frac{t_{10}}{f_0}}.$$

Using this relation, we obtain from (3.21)

$$t_{10} \frac{R_c}{k} = 2 \frac{f_0}{t_{10}} + \left(\frac{f_0}{t_{10}}\right)^2 \left[1 + \frac{1}{2} \left\{\frac{t_{10}}{f_0} \frac{b - a}{\sinh \frac{t_{10}}{f_0}}\right\}^2\right] \sinh 2 \frac{t_{10}}{f_0}. \tag{3.26}$$

Eqs. (3.25) and (3.26) can now be used to determine f_0 and t_{10}. This completes the formal determination of the three constants involved. Since the constants cannot be solved explicitly and solution multiplicity is involved, we give a qualitative discussion of the solutions in the next section.

4. Non-uniqueness of asymptotic configuration. Before proceeding it is convenient to introduce the following parameters:

$$\gamma_0 \equiv t_{10}/f_0\,, \quad \tau_{10} \equiv (R_c/k) t_{10}\,, \quad \rho \equiv a/b \leq 1, \quad u \equiv b\gamma_0\,. \tag{4.1}$$

The shape of the asymptotically stretched membrane is given by (3.15), which can now be written as

$$\gamma_0 x_0 = \cosh \gamma_0 y_0 . \tag{4.2}$$

Thus, the number of asymptotic configurations is determined by the number of values of the catenary parameter γ_0 satisfying (3.25).

The values of γ_0 satisfying (3.25) are determined by the intersections, in the $u - \gamma_0$ plane, of the line

$$\gamma_0 = (1/b)u \tag{4.3}$$

and the curve

$$Q^{\pm}[\rho]: \gamma_0 = Q^{\pm}(u, \rho) \equiv \tfrac{1}{2}(\cosh^{-1} u \pm \cosh^{-1} \rho u), \qquad u \geq 1/\rho, \tag{4.4}$$

where the notation is self-explanatory. It can be easily checked that $\partial Q^{+}/\partial u > 0$, $\partial Q^{-}/\partial u < 0$ for $u > 1/\rho$. Moreover, the two branches $Q^{\pm}$ meet and are tangent to each other at

$$u = 1/\rho, \qquad \gamma_0 = \tfrac{1}{2} \cosh^{-1} (1/\rho), \tag{4.5}$$

where $\partial Q^{\pm}/\partial u = \pm\infty$. It follows that the two branches can be combined into a single curve $Q[\rho]$ defined by

$$Q[\rho]: u = Q(\gamma_0 , \rho) \tag{4.6}$$

where Q consists of the unique inverses of $Q^{\pm}$ and $\partial Q/\partial \gamma_0 = 0$ at $\gamma_0 = \frac{1}{2} \cosh^{-1} 1/\rho$. The quantity ρ is a parameter and a set of $Q[\rho]$-curves is given in Fig. 1.

While Eqs. (4.3) and (4.4) cannot be solved explicitly, the $Q[\rho]$-curves supply a full and clear picture of the nature of the solutions involved. Let $\gamma_0 = (1/b^*)u$ be the unique tangent to the curve $Q[\rho]$ and let $u = u^*$ and $\gamma_0 = \gamma_0^*$ be the point of tangency, then

$$\frac{Q^{+}(u^*, \rho)}{u^*} = \left.\frac{\partial Q^{+}}{\partial u}\right|_{u=u^*}, \tag{4.7}$$

$$\gamma_0^* = Q^*(u^*, \rho), \qquad b^* = u^*/\gamma_0^*. \tag{4.8}$$

Eq. (4.7) determines u^* as a function of ρ and hence all the starred quantities are functions of ρ. For given ρ and b, there are the following possibilities:

(1) $b < b^*(\rho)$. The present asymptotic analysis does not yield a solution of the form (4.2).

(2) $b = b^*(\rho)$. There exists a unique catenoid as the asymptotic solution. The catenary parameter γ_0 takes the value $\gamma_0^*(\rho)$. Moreover, the catenary satisfies the conditions $y_0(0) < 0$, $y_0(1) > 0$.

(3) $b^*(\rho) < b \leq b_0(\rho) \equiv (\rho/2) \cosh^{-1} (1/\rho)$. The quantity $b_0(\rho)$ is determined by (4.5). Two solutions exist. Let $\gamma_0^{(1)}$ and $\gamma_0^{(2)}$, $\gamma_0^{(1)} < \gamma_0^{(2)}$, be the two catenary parameters. The corresponding catenaries satisfy, respectively, the conditions $y_0^{(1)}(0) \leq 0$, $y_0^{(1)}(1) > 0$ and $y_0^{(2)}(0) < 0$, $y_0^{(2)}(1) > 0$, where the equality holds only when $b = b_0(\rho)$.

(4) $b > b_0(\rho)$. Two distinct solutions exist. Let $\gamma_0^{(1)}$ and $\gamma_0^{(2)}$, $\gamma_0^{(1)} < \gamma_0^{(2)}$, be the two catenary parameters. The corresponding catenaries satisfy the conditions $y_0^{(2)}(0) < 0$, $y_0^{(2)}(1) > 0$ and $y_0^{(1)}(S) > 0$.

This completes the determination of the shape of the asymptotically stretched membrane.

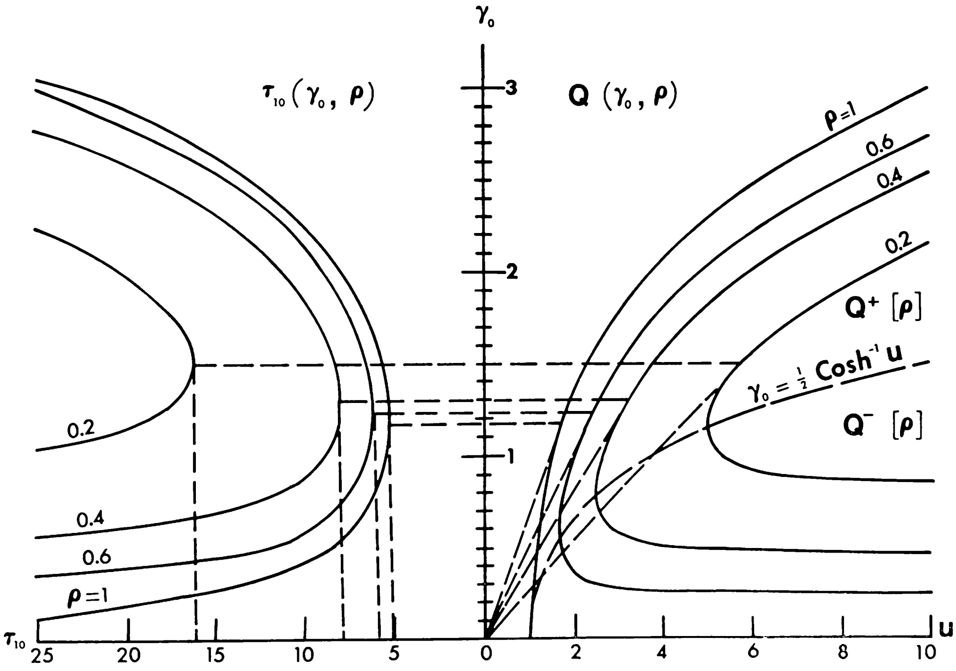

FIG. 1.

We must now determine t_{10} which determines the magnitude of the stress resultants and is also required in the $y_0 - S$ relation (3.17). The quantity t_{10} is determined by (3.26). In terms of the parameters (4.1), (3.26) is

$$\tau_{10} = \frac{2}{\gamma_0} + \frac{\sinh 2\gamma_0}{\gamma_0^2} + (1 - \rho)^2 \frac{u^2}{\gamma_0^2} \coth \gamma_0 \tag{4.9}$$

where u satisfies (4.6) and hence is a function of γ_0 and ρ. It follows that τ_{10} is a function of γ_0 and ρ. A lengthy but straightforward calculation shows that $\partial^2\tau_{10}/\partial\gamma_0^2 > 0$ and

$$\partial\tau_{10}/\partial\gamma_0 = 0 \quad \text{at} \quad \gamma_0 = \gamma_0^*. \tag{4.10}$$

A set of $\tau_{10}(\gamma_0\ , \rho)$-curves is also given in Fig. 1. We conclude the above discussion by outlining the graphic construction of the solution pertinent to Fig. 1. For a given pair of a and b, the following steps are performed:

(1) Erect a line defined by $\gamma_0 = (1/b)u$. This line and the curve $Q[\rho]$ have two intersections in general. Denote the values of γ_0 at the intersections by $\gamma_0^{(1)}(b, \rho)$ and $\gamma_0^{(2)}(b, \rho)$ where $\gamma_0^{(1)} \leq \gamma_0^{(2)}$ is assumed.

(2) Determine $\tau_{10}^{(1)}(b, \rho) \equiv \tau_{10}(\gamma_0^{(1)}, \rho)$ and $\tau_{10}^{(2)}(b, \rho) \equiv \tau_{10}(\gamma_0^{(2)}, \rho)$ from the left portion of Fig. 1.

(3) These values, together with the geometric constant R_c and the material constant k, yield the constants t_{10} and f_0 by (4.1).

(4) The constant $y_0(0)$ is determined from (3.23), i.e.,

$$y_0(0) = \mp\frac{1}{\gamma_0} \cosh^{-1} \gamma_0 a \quad \text{for} \quad \gamma_0 \gtrless \tfrac{1}{2} \cosh^{-1} \frac{1}{\rho}.$$

Finally, we make an attempt to provide certain physical explanations for the multiple solutions involved. This is done through an energy consideration. Since only displacement boundary conditions are specified, the total potential energy is the strain energy stored in the membrane. Eq. (2.16) implies that

$$w(\lambda_1, \lambda_2, \epsilon) \sim k\lambda_{10}^2\lambda_{20}^2 = \frac{t_{10}^2}{4k} = \frac{k}{4Rc^2}\tau_{10}^2 \tag{4.11}$$

It follows that the total potential energy, which equals to the total initial area $2\pi R_c$ multiplied by the constant energy density (4.11), is proportional to τ_{10}^2. If we assume that the membrane prefers the state of a lower energy then the configuration corresponding to the smaller one of $\tau_{10}^{(1)}$ and $\tau_{10}^{(2)}$ may be called the preferred state.

5. First-order correction. Differentiating (2.18)–(2.26) with respect to ϵ and then setting $\epsilon = 0$, we obtain, after using the notation (2.17),

$$\lambda_{11} = dl_1/dS, \qquad \lambda_{21} = x_1/R, \tag{5.1, 2}$$

$$t_{11} = 2k(\lambda_{10}\lambda_{21} + \lambda_{11}\lambda_{20}) + 2\frac{\lambda_{10}}{\lambda_{20}}, \tag{5.3}$$

$$t_{21} = 2k(\lambda_{10}\lambda_{21} + \lambda_{11}\lambda_{20}) + 2\frac{\lambda_{20}}{\lambda_{10}}, \tag{5.4}$$

$$x_0\frac{dt_{11}}{dS} = (t_{21} - t_{11})\frac{dx_0}{dS}, \tag{5.5}$$

$$\frac{d}{dS}(x_0t_{10}\phi_1\cos\phi_0 + x_0t_{11}\sin\phi_0 + x_1t_{10}\sin\phi_0) = 0, \tag{5.6}$$

$$-\frac{\lambda_{11}}{\lambda_{10}^2}\frac{dx_0}{dS} + \frac{1}{\lambda_{10}}\frac{dx_1}{dS} = -\phi_1\sin\phi_0, \tag{5.7}$$

$$-\frac{\lambda_{11}}{\lambda_{10}^2}\frac{dy_0}{dS} + \frac{1}{\lambda_{10}}\frac{dy_1}{dS} = \phi_1\cos\phi_0, \tag{5.8}$$

$$x_1(0) = 0, \quad x_1(1) = 0, \quad y_0(1) - y_0(0) = 0. \tag{5.9}$$

Substituting (5.3) and (5.4) into (5.5) and making other appropriate substitutions, we obtain

$$dt_{11}/dS = F(S) \tag{5.10}$$

where

$$F(S) = \frac{2}{R}\left[1 - \left(\frac{t_{10}}{2k}\right)^2\frac{R^4}{x_0^4}\right]\tanh\frac{t_{10}}{f_0}y_0. \tag{5.11}$$

Integrating (5.10) yields

$$t_{11}(S) = t_{11}(0) + \int_0^S F(S')\,dS' \tag{5.12}$$

where the constant $t_{11}(0)$ remains to be determined.

Eqs. (5.2) and (5.3) imply

$$\lambda_{11} = \frac{1}{2k}\frac{t_{11}}{\lambda_{20}} - \frac{1}{k}\frac{\lambda_{10}}{\lambda_{20}^{\ 2}} - \frac{\lambda_{10}}{\lambda_{20}}\frac{x_1}{R}. \tag{5.13}$$

Substituting (5.13) into (5.7) and using (3.1), (3.2), (3.3) and (3.8), we obtain

$$\phi_1 = -\frac{dx_1}{dy_0} - \frac{1}{x_0}\frac{dx_0}{dy_0}x_1 + \frac{1}{t_{10}}\frac{dx_0}{dy_0}t_{11} - \frac{R^2}{kx_0^{\ 2}}\frac{dx_0}{dy}. \tag{5.14}$$

Substituting (5.14) into the integrated form of (5.6) and using the zeroth-order solution repeatedly, we get

$$x_0^{\ 2}\frac{dx_0}{dy_0}\frac{dx_1}{dy_0} + x_0\left[\left(\frac{dx_0}{dy_0}\right)^2 - 1\right]x_1 - \frac{x_0^{\ 2}}{t_{1p}}\left[\left(\frac{dx_0}{dy_0}\right)^2 + 1\right]t_{11} = -\frac{f_1R}{2k}\left(\frac{dy_0}{dS}\right)^{-1} - \frac{R^2}{k}\left(\frac{dx_0}{dy_0}\right)^2 \tag{5.15}$$

or, after applying (3.15) and (3.16),

$$\cosh^2\gamma_0y_0\sinh\gamma_0y_0\frac{dx_1}{dy_0} + \frac{t_{10}}{f_0}\cosh\gamma_0y_0(\sinh^2\gamma_0y_0 - 1)x_1$$
$$- \frac{1}{t_{10}}\cosh^4\gamma_0y_0t_{11} = -\frac{f_1}{f_0}\cosh^2\frac{t_{10}}{f_0}y_0 - \frac{R^2}{k}\left(\frac{t_{10}}{f_0}\right)^2\sinh^2\frac{t_{10}}{f_0}y_0\,. \tag{5.16}$$

Dividing (5.16) by $\sinh^2\gamma_0y_0$, we get

$$\frac{d}{dy_0}\left[\frac{\cosh^2\gamma_0y_0}{\sinh\gamma_0y_0}x_1 - \frac{1}{t_{10}\gamma_0}(-\coth\gamma_0y_0 + \tfrac{1}{4}\sinh 2\gamma_0y_0 + \tfrac{3}{2}\gamma_0y_0)t_{11}\right]$$
$$= -\frac{1}{t_{10}\gamma_0}(-\coth\gamma_0y_0 + \tfrac{1}{4}\sinh 2\gamma_0y_0 + \tfrac{3}{2}\gamma_0y_0)\frac{dt_{11}}{dy_0} - \frac{f_1}{f_0}\coth^2\gamma_0y_0 - \gamma_0^{\ 2}\frac{R^2}{k} \tag{5.17}$$

or, after applying (3.16) and (5.10),

$$d\left[\frac{\cosh^2\gamma_0y_0}{\sin\gamma_0y_0}x_1 - \frac{t_{11}}{t_{10}\gamma_0}(-\coth\gamma_0y_0 + \tfrac{1}{4}\sinh 2\gamma_0y_0 + \tfrac{3}{2}\gamma_0y_0)\right.$$
$$\left. + \frac{f_1}{f_0}\left(y_0 - \frac{1}{\gamma_0}\coth\gamma_0y_0\right)\right] = G(S)\,dS \tag{5.18}$$

where

$$G(S) = -\frac{1}{t_{10}\gamma_0}(-\coth\gamma_0y_0 + \tfrac{1}{4}\sinh 2\gamma_0y_0 + \tfrac{3}{2}\gamma_0y_0)F(S) - \frac{\gamma_0^{\ 3}R^3t_{10}}{2k^2\cosh^2\gamma_0y_0}. \tag{5.19}$$

Integrating (5.18) yields

$$x_1 = \frac{t_{11}}{t_{10}\gamma_0}\left(-\frac{1}{\cosh\gamma_0y_0} + \frac{1}{2}\frac{\sinh^2\gamma_0y_0}{\cosh\gamma_0y_0} + \frac{3}{2}\gamma_0y_0\frac{\sinh\gamma_0y_0}{\cosh^2\gamma_0y_0}\right)$$
$$- \frac{f_1}{f_0}\left(y_0\frac{\sin\gamma_0y_0}{\cosh^2\gamma_0y_0} - \frac{1}{\gamma_0}\frac{1}{\cosh\gamma_0y_0}\right) + \frac{\sinh\gamma_0y_0}{\cosh^2\gamma_0y_0}\int_0^S G(S')\,dS' + c_1\frac{\sinh\gamma_0y_0}{\cosh^2\gamma_0y_0} \tag{5.20}$$

where c_1 is an unknown integration constant.

Eliminating λ_{11} from (5.7) and (5.8), using (5.14) and applying the zeroth-order solution repeatedly, we get

$$d\left[y_1 + x_1 \sinh \gamma_0 y_0 - \frac{t_{11}}{t_{10}\gamma_0}\left(\tfrac{1}{4}\sinh 2\gamma_0 y_0 + \tfrac{1}{2}\gamma_0 y_0\right)\right] = H(S)\, dS \tag{5.21}$$

where

$$H(S) = -\frac{1}{t_{10}\gamma_0}\left(\tfrac{1}{4}\sinh 2\gamma_0 y_0 + \tfrac{1}{2}\gamma_0 y_0\right)F(S) - \frac{\gamma_0^3 R^3 t_{10}}{2k^2 \cosh^2 \gamma_0 y_0}. \tag{5.22}$$

Integrating (5.21) yields

$$y_1 = -x_1 \sinh \gamma_0 y_0 + \frac{t_{11}}{t_{10}\gamma_0}\left(\tfrac{1}{4}\sinh 2\gamma_0 y_0 + \tfrac{1}{2}\gamma_0 y_0\right) + \int_0^S H(S')\, dS' + d_1 \tag{5.23}$$

where d_1 is an integration constant. Since d_1 represents only a rigid-body displacement along the z-axis, we conveniently set

$$d_1 = -\frac{t_{11}(0)}{t_{10}\gamma_0}\left[\tfrac{1}{4}\sinh 2\gamma_0 y_0(0) + \tfrac{1}{2}\gamma_0 y_0(0)\right]. \tag{5.24}$$

We must now determine the three constants $t_{11}(0)$, f_1 and c_1 in such a way that the three conditions (5.9) are satisfied. The last of (5.9), together with the first two of (5.9) and the relation (3.21), yields

$$t_{11}(0) = -\frac{2k}{\gamma_0 R_c}\left\{\left[\tfrac{1}{4}\sinh 2\gamma_0 y_0(1) + \tfrac{1}{2}\gamma_0 y_0(1)\right]\int_0^1 F(S)\, dS + \int_0^1 H(S)\, dS\right\}. \tag{5.25}$$

Finally, the first two conditions of (5.9) yield

$$\begin{aligned} f_1 = f_0\Bigg\{&\frac{t_{11}(0)}{t_{11}\gamma_0}\left[\frac{1}{2}\tau_{10}\gamma_0^2 + 2\gamma_0 - \frac{\sinh 2\gamma_0}{\sinh \gamma_0 y_0(0) \sinh \gamma_0 y_0(1)}\right] \\ &+ \frac{1}{t_{10}\gamma_0}\left[-\frac{\cosh \gamma_0 y_0(1)}{\sinh \gamma_0 y_0(1)} + \tfrac{1}{4}\sinh 2\gamma_0 y_0(1) + \tfrac{3}{2}\gamma_0 y_0(1)\right]\int_0^1 F(S)\, dS \\ &+ \int_0^1 G(S)\, dS \div \left\{\frac{\sinh 2\gamma_0}{\gamma_0 \sinh \gamma_0 y_0(0) \sinh \gamma_0 y_0(1)} - 2\right\}, \end{aligned} \tag{5.26}$$

$$\begin{aligned} c_1 = -\frac{t_{11}(0)}{t_{11}\gamma_0}&\left[-\frac{\cosh \gamma_0 y_0(0)}{\sinh \gamma_0 y_0(0)} + \tfrac{1}{4}\sinh 2\gamma_0 y_0(0) + \tfrac{3}{2}\gamma_0 y_0(0)\right] \\ &+ \frac{f_1}{f_0}\left[y_0(0) - \frac{1}{\gamma_0}\frac{\cosh \gamma_0 y_0(0)}{\sinh \gamma_0 y_0(0)}\right]. \end{aligned} \tag{5.27}$$

The function t_{21} can be conveniently calculated from (5.5). We have

$$t_{21} = t_{11} + \frac{4k}{\gamma_0^2 t_{10}}\frac{\cosh^2 \gamma_0 y_0}{R^2}\left[1 - \left(\frac{t_{10}}{2k}\right)^2 \frac{R^4}{\gamma_0^4}\right]. \tag{5.28}$$

The other quantities can be computed accordingly but are not given here.

6. Stretching of a cylindrical membrane as an example. Consider a cylindrical membrane characterized by a meridian curve

$$C:\ \begin{array}{l} r = R(S) \equiv \tfrac{1}{2} \\ z = Z(S) \equiv S - \tfrac{1}{2} \end{array}, \qquad 0 \le S \le 1. \tag{6.1}$$

The membrane is stretched axially and radially so that the meridian curve of the equilibrium configuration defined by

$$c: \begin{array}{l} r = X(S) \\ z = Y(S) \end{array} \tag{6.2}$$

satisfies the conditions

$$\begin{aligned} X(0) &= X(1) = \epsilon^{-1/2}2 \\ Y(1) - Y(0) &= \epsilon^{-1/2}2 \end{aligned} \tag{6.3}$$

where ϵ is a small parameter.

In terms of the notation (4.1), we have $a = b = 2$ and $\rho = 1$. Eqs. (3.25) and (3.26) yield:

$$\gamma_0 = \begin{array}{l} \gamma_0^{(1)} = 0.5890 \\ \gamma_0^{(2)} = 2.150 \end{array} \qquad \tau_{10} = \begin{array}{l} \tau_{10}^{(1)} = 7.6328 \\ \tau_{10}^{(2)} = 8.7090 \end{array} \tag{6.4}$$

The corresponding asymptotic solutions are:

$$x(S) = \frac{1}{\gamma_0^{(n)}} \cosh \gamma_0^{(n)} y(S), \tag{6.5}$$

$$S = \frac{1}{\tau_{10}^{(n)} \gamma_0^{(n)2}} \{\tfrac{1}{2}[\sinh 2\gamma_0^{(n)} y + \sinh 2\gamma_0^{(n)}] + \gamma_0^{(n)}(y + 1)\}, \tag{6.6}$$

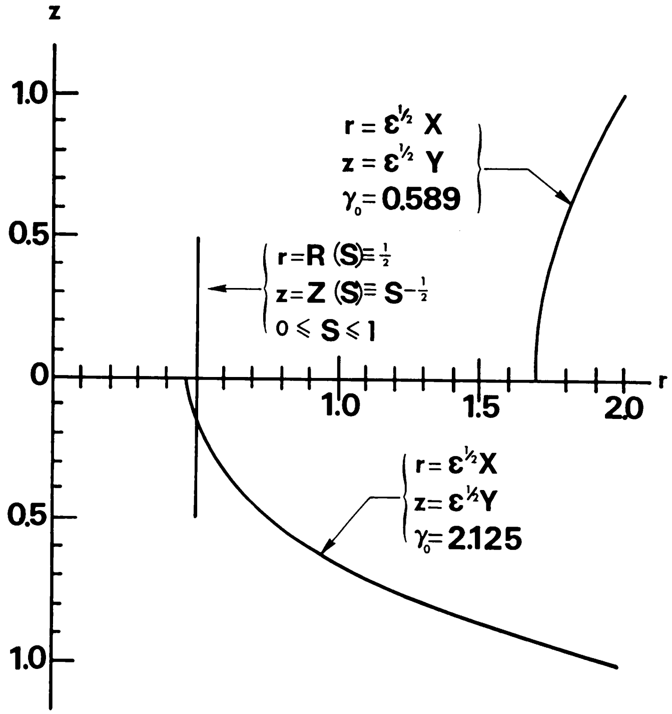

FIG. 2.

where $n = 1, 2$ and $\epsilon^{1/2}X \sim x$, $\epsilon^{1/2}Y \sim y$. The asymptotic stress resultants are

$$\epsilon T_1 \sim \epsilon T_2 \sim 2k\tau_{10}^{(n)}, \qquad n = 1, 2. \tag{6.7}$$

The two solutions are plotted in Fig. 2.

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