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PLANE WAVES AND STABILITY OF ELASTIC PLATES*

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1. Introduction. Within the context of the linear theory of thin, homogeneous elastic plates, we derived [1] a relatively simple condition for the reality of plane wave frequencies corresponding to real wave vectors. When this condition is satisfied, a plate of arbitrary size and shape, with edges rigidly clamped, is in stable static equilibrium, according to a static energy criterion for stability. Our aim is to demonstrate this. An analogous result for three-dimensional elasticity theory is derived by van Hove [2], and we use an adaptation of his techniques.

A condition similar to but somewhat weaker than that here employed suffices to guarantee this stability for plates of sufficiently small size, as follows from the analysis of van Hove [2, Sec. 3]. It is easily seen that this size restriction is indispensable. It seems possible that the condition here used is necessary for plates of all sizes and shapes to be stable, but we do not establish this.

2. Basic equations. Roughly, we consider equations of the theory of Cosserat plates, linearized about static equilibrium configurations which are homogeneous, subject to zero loads on their faces, the character of the edge loading being unspecified. In this configuration they are to have a planar form, and are referred to plane rectangular Cartesian material coordinates (u^1, u^2). In fact, this includes the possibility that their natural states are of the form of right circular cylinders which have been pulled into planar form by some edge loadings. Formally, we employ the linearized equations used by Ericksen [1, 3], *viz.*

$$2\chi = \mathbf{V}_{,\alpha}\cdot\mathcal{M}^{\alpha\beta}\mathbf{V}_{\beta} + 2\mathbf{V}\cdot\mathcal{M}^{\alpha}\mathbf{V}_{,\alpha} - \mathbf{V}\cdot\mathcal{L}\mathbf{V}, \tag{2.1}$$

$$\left(\frac{\partial\chi}{\partial\mathbf{V}_{,\alpha}}\right)_{,\alpha} - \frac{\partial\chi}{\partial\mathbf{V}} = \mathcal{K}\ddot{\mathbf{V}},$$

$$= \mathcal{L}^{\alpha\beta}\mathbf{V}_{,\alpha\beta} + \mathcal{L}^{\alpha}\mathbf{V}_{,\alpha} + \mathcal{L}\mathbf{V}. \tag{2.3}$$

wherein χ represents the usual quadratic approximation to the stored energy per unit reference area. Here $\mathbf{V}$ is a six-vector, incorporating the displacement of the reference surface and a measure of deformation in the thickness direction. The $\mathcal{L}$'s, $\mathcal{M}$'s and $\mathcal{K}$ are 6×6 constant matrices, with $\mathcal{K}$ positive definite, such that

$$\begin{gathered} \mathcal{K} = \mathcal{K}^T > 0, \qquad \mathcal{M}^{\alpha\beta} = \mathcal{M}^{\beta\alpha T} \qquad \mathcal{L} = \mathcal{L}^T, \\ 2\mathcal{L}^{\alpha\beta} = \mathcal{M}^{\alpha\beta} + \mathcal{M}^{\beta\alpha} = 2\mathcal{L}^{\beta\alpha} = 2\mathcal{L}^{\alpha\beta T}, \\ \mathcal{L}^{\alpha} = \mathcal{M}^{\alpha T} - \mathcal{M}^{\alpha} = -\mathcal{L}^{\alpha T}. \end{gathered} \tag{2.4}$$

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In [1, 3], we examined complex plane wave solutions of (2.3), of the form

$$\mathbf{V} = \mathbf{A} \exp i(k_\alpha u^\alpha - \omega t), \tag{2.5}$$

wherein $\mathbf{A}$, k_α and ω are constants, k_α being assumed real. In [1], we showed that, for all possible ω to be real, it is necessary and sufficient that, for all complex six-vectors $\mathfrak{B}$, and all real k_α ,

$$\mathfrak{B} \cdot \mathcal{H} \mathfrak{B}^* \geqq 0. \tag{2.6}$$

Here, asterisks denote complex conjugates and $\mathcal{H}$ is the Hermetian matrix

$$\mathcal{H} = \mathcal{L}^{\alpha\beta} k_\alpha k_\beta - i\mathcal{L}^\alpha k_\alpha - \mathcal{L}. \tag{2.7}$$

Equivalently, since k_α is real, (2.6) can be rewritten as

$$i\mathfrak{B} k_\alpha \cdot \mathcal{L}^{\alpha\beta}(i\mathfrak{B} k_\alpha)^* + \mathfrak{B} \cdot \mathcal{L}^\alpha (i\mathfrak{B} k_\alpha)^* - \mathfrak{B} \cdot \mathcal{L} \mathfrak{B}^* \geqq 0, \tag{2.8}$$

hereafter called the *hyperellipticity condition.* In particular, (2.8) implies the strong ellipticity condition; for all real $\mathbf{b}$ and k_α ,

$$\mathbf{b} \cdot \mathcal{L}^{\alpha\beta} k_\alpha k_\beta \mathbf{b} = \mathbf{b} \cdot \mathfrak{M}^{\alpha\beta} k_\alpha k_\beta \mathbf{b} \geqq 0, \tag{2.9}$$

the condition employed by van Hove [2]. Obviously, (2.9) does not imply (2.8).

3. Stability. We now consider a plate, in a loading device of the type described earlier, which rigidly clamps its edges. Then, as is discussed by Ericksen [4], for example, the energy criterion reads

$$\int_D \chi \, du^1 \, du^2 \geqq 0, \tag{3.1}$$

where χ, given by (2.1), is evaluated for any virtual displacement $\mathbf{V}$ such that

$$\mathbf{V} = 0 \quad \text{on} \quad \partial D. \tag{3.2}$$

Below, we refer to this type of stability as *Hadamard stability*. Here D, the domain occupied by the plate in the reference configuration, is assumed to be bounded and smooth enough for the divergence theorem to hold. Virtual displacements are to be continuous on the closure of D, with square-integrable partial derivations. Extension of ensuing analyses to unbounded domains involves only minor complications. Now, following van Hove [2], we extend the domain of such $\mathbf{V}$ to the entire plane by setting

$$\mathbf{V} = 0 \quad \text{outside} \quad D, \tag{3.3}$$

and introduce the Fourier transforms

$$\hat{\mathbf{V}}(k_\alpha) = \frac{1}{2\pi} \int_D \mathbf{V} \exp(ik_\alpha u^\alpha) \, du^1 \, du^2, \; - ik_\beta \hat{\mathbf{V}} = \frac{1}{2\pi} \int_D \mathbf{V}_{,\beta} \exp(ik_\alpha u^\alpha) \, du^1 \, du^2. \tag{3.4}$$

Then, by Parseval's theorem, with $\mathbf{V}$ real, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\mathbf{V}} \otimes \hat{\mathbf{V}}^* \, dk_1 \, dk_2 = \int_D \mathbf{V} \otimes \mathbf{V} \, du^1 \, du^2, \tag{3.5}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\mathbf{V}} \otimes (-\mathbf{V} ik_\beta)^* \, dk_1 \, dk_2 = \int_D \mathbf{V} \otimes \mathbf{V}_{,\beta} \, du^1 \, du^2, = -\int_D \mathbf{V}_{,\beta} \otimes \mathbf{V} \, du^1 \, du^2, \tag{3.6}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (i\hat{\mathbf{V}}k_\alpha) \otimes (i\hat{\mathbf{V}}k_\beta)^* \, dk_1 \, dk_2 = \int_D \mathbf{V}_{,\alpha} \otimes \mathbf{V}_{,\beta} \, du^1 \, du^2 = \int_D \mathbf{V}_{,\beta} \otimes \mathbf{V}_{,\alpha} \, du^1 \, du^2. \quad (3.7)$$

The alternative evaluations occurring on the right follow from symmetries evident on the left and uniqueness of Fourier transforms. Using them and (2.4), we can rewrite (3.1) as

$$\int_D \chi \, du^1 \, du^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [i\hat{\mathbf{V}}k_\alpha \cdot \mathfrak{L}^{\alpha\beta}(i\hat{\mathbf{V}}k_\beta)^* + \hat{\mathbf{V}} \cdot \mathfrak{L}^\alpha(ik_\alpha\hat{\mathbf{V}})^* - \hat{\mathbf{V}} \cdot \mathfrak{L}\hat{\mathbf{V}}]^* \, du^1 \, du^2. \quad (3.8)$$

It is then obvious that (2.8) implies (3.1). Thus, *hyperellipticity implies Hadamard stability*. Basically, this is what we set out to demonstrate, but a few comments might be in order.

First, because of the Galilean invariance normally assumed in mechanics, the inequality (2.8) cannot be strict when $k_\alpha = 0$; the plane waves then include among them rigid translations. The latter correspond to certain non-zero choices of $\mathfrak{B}$ for which equality holds in (2.8). Fortunately, the point $k_\alpha = 0$ in $\mathbf{k}$-space is a set of measure zero. Also, (3.2) rules out non-trivial rigid translations. It then follows that, if the inequality (2.8) is strict for $k_\alpha \neq 0$, then the inequality (3.1) will be strict. The strict inequality is, of course, what is needed for a classical Kirchoff-type proof of uniqueness of static solutions, with $\mathbf{V}$ specified on the boundary. It also has some importance relative to the frequencies of normal modes of vibration, corresponding to boundary conditions of the type (3.2); such frequencies will be real if (3.1) holds, as is discussed by Ericksen [4]. However, the lowest frequency can be zero if (3.1) is not strict, a sign of impending instability.

If only the strong ellipticity condition (2.9) holds, but in the strict sense, one can use the Schwarz and Poincaré inequalities to show that (3.1) holds, provided that the dimensions of D are sufficiently small. In essence, this observation is due to van Hove [2, Sec. 3]. For a specified region of larger size, it would thus appear that some condition intermediate between (2.8) and (2.9) would suffice. As is now well known, (2.9) is a necessary condition that (3.1) hold.

References

[1] J. L. Ericksen, *Plane infinitesinal waves in homogeneous elastic plates*, J. Elasticity **3**, 161–167 (1973)

[2] Leon van Hove, *Sur l'extension de la condition de Legendre du calcul des variations aux intégrales multiples à plusierers fonctions inconnues*, Proc. Kon. Ned. Akad. Wet. **50**, 18–23 (1947)

[3] J. L. Ericksen, *Simpler problems for elastic plates*, Int. J. Solids Structures **9**, 141–149 (1973)

[4] J. L. Ericksen, *Wave propagation in thin elastic shells*, Arch. Ratl. Mech. Anal. **43**, 167–178 (1971)