QUARTERLY OF APPLIED MATHEMATICS
JULY, 1974

PLANE WAVES IN LINEAR VISCOELASTIC MATERIALS*

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1. Introduction. In this paper we discuss the propagation of plane sinusoidal waves in linear viscoelastic materials, both anisotropic and isotropic. Unlike the usual discussions (see, for example, [1]), we do not here assume that the planes of constant amplitude and constant phase are parallel. We do, however, assume that the imaginary parts of the complex moduli are small compared with their real parts and that correspondingly the magnitude of the imaginary part of the slowness vector is small compared with that of the real part. This implies that the attenuation of the wave is small in distances of travel of the order of a wavelength.

For a general anisotropic material it is found that, provided the imaginary parts of the complex moduli are sufficiently small compared with their real parts, for any specified directions of the normals to the planes of constant phase and constant amplitude, except for those satisfying a certain relation which depends on the real part of the complex modulus tensor, three waves can be propagated. Each of these is slightly elliptically polarized, the major axis of the ellipse being large compared with its minor axis.

In the case of an isotropic material, one of these waves is nearly longitudinal and the others are nearly transverse waves. The nearly longitudinal wave is slightly elliptically polarized with its major axis in the longitudinal direction and its minor axis coplanar with the normals to the planes of constant amplitude and constant phase. The nearly transverse waves are also slightly elliptically polarized with their major axes arbitrarily oriented in the planes of constant phase and their minor axes in the direction of propagation. For these waves, both the real and imaginary parts of the complex slowness vector are independent of the orientation of the direction, in the transverse plane, of the major axis of the ellipse. Consequently, waves may also be propagated which result from the superposition of two such nearly transverse waves, for which the displacements in the transverse plane have different directions of polarization and different phases.

In an isotropic viscoelastic material, for which the imaginary parts of the Lamé constants are sufficiently small compared with their real parts, elliptically-polarized nearly longitudinal waves and elliptically-polarized nearly transverse waves of the type described can be propagated for any inclination, other than a right angle, of the normals to the planes of constant phase and constant amplitude. On the other hand, in an iso-

* Received January 24, 1973. This work was carried out with the support of a grant (GP-30904XI) from the National Science Foundation to Lehigh University.

tropic elastic material, plane waves can only be propagated if the planes of constant phase are either parallel or perpendicular to planes of constant amplitude.

Lockett [2] has considered the propagation, in an isotropic linear viscoelastic material, of plane sinusoidal waves for which the planes of constant amplitude are not necessarily parallel to the planes of constant phase and has discussed the reflection-refraction problem for such waves. In his discussion he draws attention to the elliptical character of the polarization of the waves when the planes of constant phase and constant amplitude are not parallel. Lockett has not, however, made the assumptions which we have made here that the imaginary parts of the Lamé constants are small compared with the real parts and consequently his results are, in some respects, less explicit than ours. Also, he restricts his analysis to the case when the polarization is entirely in the plane formed by the normals to the planes of constant phase and constant amplitude.

2. Anisotropic materials. We consider a linear viscoelastic material undergoing a deformation for which the components of the displacement vector $\mathbf{u}(\tau)$ at time τ, in a rectangular cartesian coordinate system x, are $u_i(\tau)$. It is assumed that the components σ_{ij} of the stress tensor $\boldsymbol{\sigma}$, at time t, are given by

$$\sigma_{ij} = C_{ijkl}e_{kl}(t) + \int_{-\infty}^{t} f_{ijkl}(t - \tau)e_{kl}(\tau)\, d\tau, \tag{2.1}$$

where $e_{kl}(\tau)$ are the components of the infinitesimal strain tensor $\mathbf{e}(\tau)$ at time τ, given by

$$e_{kl}(\tau) = \frac{1}{2}\left[\frac{\partial u_k(\tau)}{\partial x_l} + \frac{\partial u_l(\tau)}{\partial x_k}\right]. \tag{2.2}$$

C_{ijkl} and f_{ijkl} are symmetric with respect to interchange of i and j and of k and l.

We now suppose that the deformation corresponds to a damped plane sinusoidal wave of angular frequency ω. Using the usual complex notation, we can write the complex displacement $u_i(\tau)$ in the form

$$u_i(\tau) = U_i \exp \iota\omega(S_k x_k - \tau), \tag{2.3}$$

where U_i is a complex constant vector and S_k are the components in the system x of the complex slowness vector $\mathbf{S}$.

Introducing (2.3) and (2.2) into (2.1), we see that the complex stress is given by

$$\sigma_{ij} = \Sigma_{ij} \exp \iota\omega(S_k x_k - t), \tag{2.4}$$

where

$$\Sigma_{ij} = \iota\omega c_{ijkl} S_l U_k \tag{2.5}$$

and

$$c_{ijkl} = c_{ijkl}(\omega) = \left[C_{ijkl} + \int_{-\infty}^{t} f_{ijkl}(t - \tau) \exp \iota\omega(t - \tau)\, d\tau\right]. \tag{2.6}$$

c_{ijkl} is the complex modulus tensor for the material.

In the absence of body forces, the stress σ_{ij} must satisfy the equations of motion

$$\partial\sigma_{ij}/\partial x_j = \rho(\partial^2 u_i/\partial t^2), \tag{2.7}$$

where $u_i = u_i(t)$ and ρ denotes the mass density of the material. Introducing (2.3), (2.4) and (2.5) into this equation, we obtain

$$(c_{ijkl}S_jS_l - \rho\delta_{ik})U_k = 0. \tag{2.8}$$

This has a non-trivial solution for U_k if and only if

$$|c_{ijkl}S_jS_l - \rho\delta_{ik}| = 0. \tag{2.9}$$

We shall assume that the planes of constant phase and the planes of constant amplitude for the wave are not necessarily the same and that they are normal to the real vectors $\mathbf{m}$ and $\mathbf{n}$ respectively, which have components m_i and n_i respectively in the system x. We shall also assume that, for the wave considered, the imaginary part of $\mathbf{S}$ is small in magnitude compared with the real part and that the imaginary part of c_{ijkl} is small compared with the real part. We may then write

$$\begin{gathered} \mathbf{S} = \mathbf{S}^+ + \iota\epsilon\mathbf{S}^- = S^+\mathbf{m} + \iota\epsilon S^-\mathbf{n} \\ c_{ijkl} = c_{ijkl}{}^+ + \iota\epsilon c_{ijkl}{}^-, \end{gathered} \tag{2.10}$$

where ϵ is small and real. We assume that the elastic part $c_{ijkl}{}^+$ of the complex modulus satisfies the Onsager relations

$$c_{ijkl}{}^+ = c_{klij}{}^+. \tag{2.11}$$

It follows from (2.8) that we may, without loss of generality, take the imaginary part of $\mathbf{U}$ to be $O(\epsilon)$. We may therefore write

$$\mathbf{U} = \mathbf{U}^+ + \iota\epsilon\mathbf{U}^-. \tag{2.12}$$

Introducing (2.10) and (2.12) into (2.8), we obtain, by equating the coefficients of ϵ^0 and ϵ separately to zero, the propagation conditions

$$\begin{gathered} \{(S^+)^2c_{ijkl}{}^+m_lm_j - \rho\delta_{ik}\}U_k{}^+ = 0, \\ S^+\{S^-c_{ijkl}{}^+(m_ln_j + m_jn_l) + S^+c_{ijkl}{}^-m_lm_j\}U_k{}^+ = -\{(S^+)^2c_{ijkl}{}^+m_lm_j - \rho\delta_{ik}\}U_k{}^-. \end{gathered} \tag{2.13}$$

3. The energy flux vector. The energy flux vector $R_i(t)$ at time t is defined as the rate at which energy leaves the material across an element of area normal to the x_i-axis, measured per unit area. It is given by

$$R_i(t) = -\sigma_{ji}{}^+(t)\dot{u}_j{}^+(t). \tag{3.1}$$

This fluctuates with time and accordingly we define a mean energy flux vector $\tilde{\mathbf{R}}$ as the average of $\mathbf{R}(t)$ over a cycle. Thus

$$\tilde{R}_i = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} R_i(t)\,dt. \tag{3.2}$$

Inserting (2.3) and (2.4) in (3.1), we find from (3.2) that

$$\tilde{R}_i = \tfrac{1}{2}\omega^2(c_{ijkl}S_lU_k\bar{U}_j)^+ \exp\,(-2\epsilon\omega S_p{}^-x_p), \tag{3.3}$$

where the bar denotes the complex conjugate. By using (2.10), (2.11) and (2.12), Eq. (3.3) becomes

$$\bar{R}_i = \tfrac{1}{2}\omega^2 S^+ c_{ijkl}{}^+ m_l U_k{}^+ U_j{}^+ \exp\,(-2\epsilon\omega S_p{}^- x_p), \tag{3.4}$$

neglecting terms of first and higher orders in ϵ but bearing in mind that x_p may be large. We note, using (2.11), that

$$\begin{aligned}\bar{R}_i n_i &= \tfrac{1}{2}\omega^2 S^+ c_{ijkl}{}^+ n_i m_l U_k{}^+ U_j{}^+ \exp\,(-2\epsilon\omega S_p{}^- x_p)\\ &= \tfrac{1}{2}\omega^2 S^+ c_{ijkl}{}^+ m_l n_j U_i{}^+ U_k{}^+ \exp\,(-2\epsilon\omega S_p{}^- x_p).\end{aligned} \tag{3.5}$$

The stress power $\mathfrak{D}$ is given by

$$\mathfrak{D} = \sigma_{ji}{}^+ (\partial \dot{u}_i{}^+/\partial x_j) \tag{3.6}$$

Thus if $\bar{\mathfrak{D}}$ denotes the amount of energy dissipated in a cycle, per unit volume, we have

$$\bar{\mathfrak{D}} = \int_0^{2\pi/\omega} \sigma_{ji}{}^+ \frac{\partial \dot{u}_i{}^+}{\partial x_j}\, dt. \tag{3.7}$$

Using (2.3) and (2.4) in (3.7), we find

$$\bar{\mathfrak{D}} = -\pi\omega^2 (c_{ijkl} S_l U_k \bar{U}_i \bar{S}_j)^- \exp\,(-2\epsilon\omega S_p{}^- x_p). \tag{3.8}$$

To the first order in ϵ, we obtain, with (2.11),

$$\bar{\mathfrak{D}} = -\epsilon\pi\omega^2 (S^+)^2 c_{ijkl}{}^- m_l m_j U_i{}^+ U_k{}^+ \exp\,(-2\epsilon\omega S_p{}^- x_p). \tag{3.9}$$

4. The propagation conditions. We now consider the propagation conditions (2.13) in detail.

If the unit vector $\mathbf{m}$ is specified, S^+ may be obtained from the secular equation

$$|(S^+)^2 c_{ijkl}{}^+ m_l m_j - \rho\delta_{ik}| = 0, \tag{4.1}$$

which is the condition that $(2.13)_1$ have a solution other than $U_k{}^+ = 0$. From (2.11) it follows that

$$c_{ijkl}{}^+ m_l m_j = c_{kjil}{}^+ m_l m_j\ , \tag{4.2}$$

and hence the three roots of (4.1) for $(S^+)^2$ are all real. To each of these roots there corresponds a solution of $(2.13)_1$ for $U_k{}^+$ which is determined uniquely in direction but not in magnitude. These three directions are, of course, mutually perpendicular.

We now multiply $(2.13)_2$ by $U_i{}^+$. Then using (2.11), $(2.13)_1$ and (4.2), and assuming $S^+ \neq 0$, we obtain

$$2S^- c_{ijkl}{}^+ m_l n_j U_i{}^+ U_k{}^+ = -S^+ c_{ijkl}{}^- m_l m_j U_i{}^+ U_k{}^+. \tag{4.3}$$

We can eliminate $U_i{}^+ U_k{}^+$ from Eq. (4.3) in the following manner. Let λ_{ik} denote the cofactor of $(S^+)^2 c_{ijkl} m_l m_j - \rho\delta_{ik}$ in det $|(S^+)^2 c_{ijkl} m_l m_j - \rho\delta_{ik}|$. Then, following Synge [2], we can express λ_{ik} in the form

$$\lambda_{ik} = \Theta U_i{}^+ U_k{}^+, \tag{4.4}$$

where Θ is a constant. Accordingly (4.3) may be rewritten as

$$2S^- c_{ijkl}{}^+ m_l n_j \lambda_{ik} = -S^+ c_{ijkl}{}^- m_l m_j \lambda_{ik}\ . \tag{4.5}$$

Eq. (4.3) determines S^- in terms of S^+, $\mathbf{m}$ and $\mathbf{n}$ provided that

$$c_{ijkl}{}^+ m_l n_j U_i{}^+ U_k{}^+ \neq 0. \tag{4.6}$$

Since $\mathbf{U}^+$ satisfies $(2.13)_1$, it can be shown that the component of $\mathbf{U}^-$ in the direction of $\mathbf{U}^+$ is not determined by $(2.13)_2$. In order to do this, we write $\mathbf{U}^-$ in the form

$$\mathbf{U}^- = \alpha\mathbf{U}^+ + \mathbf{P}, \tag{4.7}$$

where $\mathbf{P}$ is a real vector perpendicular to $\mathbf{U}^+$ and lying in the plane of $\mathbf{U}^-$ and $\mathbf{U}^+$, so that

$$\mathbf{P}\cdot\mathbf{U}^+ = 0. \tag{4.8}$$

Introducing (4.7) into $(2.13)_2$ and using $(2.13)_1$, we obtain

$$S^+\{S^-c_{ijkl}{}^+(m_ln_j + m_jn_l) + S^+c_{ijkl}{}^-m_lm_j\}U_k{}^+ = -\{(S^+)^2c_{ijkl}{}^+m_lm_j - \rho\delta_{ik}\}P_k\,. \tag{4.9}$$

Thus, Eq. $(2.13)_2$, which appears superficially to be an equation for the determination of $\mathbf{U}^-$ if $\mathbf{U}^+$, $\mathbf{m}$ and $\mathbf{n}$ are known, in fact determines only the component of $\mathbf{U}^-$ in the plane perpendicular to $\mathbf{U}^+$. This leaves the component of $\mathbf{U}^-$ parallel to $\mathbf{U}^+$, and hence the phase of the component of $\mathbf{U}$ in this direction, undetermined.

Suppose now that we have chosen $\mathbf{m}$ and solved (4.1) for S^+ and $(2.13)_1$ for $\mathbf{U}^+$. Then there is a whole plane of directions $\mathbf{n}$ for which (4.6) is not satisfied, i.e. for which

$$c_{ijkl}{}^+m_ln_jU_i{}^+U_k{}^+ = 0. \tag{4.10}$$

In general, however, the corresponding wave

$$u_i = (U_i{}^+ + \iota\epsilon U_i{}^-)\exp \iota\omega(S^+m_px_p - t)\exp(-\epsilon S^-n_px_p) \tag{4.11}$$

does not propagate in the material. For, if (4.10) holds, then, from (4.3),

$$S^+c_{ijkl}{}^-m_lm_jU_i{}^+U_k{}^+ = 0 \tag{4.12}$$

must be satisfied. Here $\mathbf{m}$ and $\mathbf{U}^+$ are assumed known so that in general (4.12) cannot be satisfied unless $S^+ = 0$, in which case the wave does not progress.

From (3.5) it is seen that (4.10) may be written

$$\tilde{R}_in_i = 0, \tag{4.13}$$

so that the mean energy flux vector lies in the plane of constant amplitude, whilst by (3.9) Eq. (4.12) may be written

$$\tilde{\mathfrak{D}} = 0, \tag{4.14}$$

so that no energy is dissipated. So, if for some choice of $\mathbf{m}$ and corresponding solutions S^+ of (4.1) and $\mathbf{U}^+$ of $(2.13)_1$ $\mathbf{n}$ is chosen to satisfy (4.10) and if (4.12) holds, then the material behaves elastically for the corresponding wave.

Of course, if (4.10) holds, then (4.3) may not be used to determine S^-. In fact S^- is arbitrary and must be determined from the boundary conditions. This arbitrariness in S^- leads to a corresponding arbitrariness in P_k, from (4.9).

The indeterminacy of $\mathbf{U}$ is, in fact, more apparent than real. We note, from (2.8), that for a specified $\mathbf{S}$ satisfying (2.9), $\mathbf{U}$ is determined apart from an arbitrary scalar multiplier. The apparent indeterminacy of the component $\alpha\mathbf{U}^+$ of $\mathbf{U}^-$ (see (4.7)) results from the approximation in which terms of order ϵ^2 are neglected. To the order ϵ, $\mathbf{P}$ is unaltered by small changes of phase in the component of $\mathbf{U}^-$ parallel to $\mathbf{U}^+$, but it would be altered if terms of higher order in ϵ were retained.

The fact that the component of $\mathbf{U}$ perpendicular to $\mathbf{U}^+$ is not zero implies that the wave is slightly elliptically polarized, the major axis of the ellipse being substantially in the direction of $\mathbf{U}^+$.

For an elastic material $c_{ijkl}^- = 0$, and from (4.3) we obtain

$$S^- = 0, \tag{4.15}$$

or

$$c_{ijkl}{}^+ m_l n_j U_i{}^+ U_k{}^+ = 0. \tag{4.16}$$

If $S^- = 0$, we have, from (4.9), $P_k = 0$, so that $\mathbf{U}^-$ and $\mathbf{U}^+$ are parallel. We note that in general Eq. $(2.13)_1$ yields three possible directions for $\mathbf{U}^+$, corresponding to the three possible solutions of (4.1) for $(S^+)^2$. These are the directions of the three mutually-perpendicular eigenvectors of the symmetric matrix $\|(S^+)^2 c_{ijkl}{}^+ m_l m_j - \rho\delta_{ik}\|$. The corresponding waves have, of course, their complex displacements linearly polarized in the directions of these eigenvectors. The arbitrariness in the choice of $\mathbf{U}^-$ corresponds to an arbitrariness in the phase of the wave.

If (4.16) is applicable, then for each $\mathbf{m}$ and corresponding $(S^+)^2$ and $\mathbf{U}^+$ this relation determines the directions of $\mathbf{n}$ for which waves *can* be propagated in an elastic material. S^- is arbitrary and must be determined from the boundary conditions.

5. Isotropic materials. If the material is isotropic, we can write the complex modulus tensor c_{ijkl} in the form

$$c_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + \lambda\delta_{ij}\delta_{kl}\,, \tag{5.1}$$

where μ and λ are the complex Lamé constants for the material. We introduce the notation

$$\mu = \mu^+ + \iota\epsilon\mu^-, \quad \lambda = \lambda^+ + \iota\epsilon\lambda^-. \tag{5.2}$$

Eqs. (2.13) then become

$$\begin{aligned} (S^+)^2[(\mu^+ + \lambda^+)(\mathbf{U}^+\cdot\mathbf{m})\mathbf{m} + \mu^+\mathbf{U}^+] - \rho\mathbf{U}^+ &= \mathbf{0}, \\ S^+S^-[2\mu^+(\mathbf{m}\cdot\mathbf{n})\mathbf{U}^+ + (\mu^+ + \lambda^+)\{(\mathbf{U}^+\cdot\mathbf{m})\mathbf{n} + (\mathbf{U}^+\cdot\mathbf{n})\mathbf{m}\}] & \\ + (S^+)^2[\mu^-\mathbf{U}^+ + (\mu^- + \lambda^-)(\mathbf{U}^+\cdot\mathbf{m})\mathbf{m}] & \\ = [\rho - \mu^+(S^+)^2]\mathbf{U}^- - (\mu^+ + \lambda^+)(S^+)^2(\mathbf{U}^-\cdot\mathbf{m})\mathbf{m}&. \end{aligned} \tag{5.3}$$

Also, Eq. (4.1) becomes

$$|(S^+)^2[(\mu^+ + \lambda^+)m_i m_k + \mu^+\delta_{ik}] - \rho\delta_{ik}| = 0. \tag{5.4}$$

By using (3.9), the dissipation per cycle, per unit volume, is given by

$$\tilde{\mathfrak{D}} = -\epsilon\pi\omega^2(S^+)^2\{\mu^-\mathbf{U}^+\cdot\mathbf{U}^+ + (\lambda^- + \mu^-)(\mathbf{U}^+\cdot\mathbf{m})^2\}\exp(-2\epsilon\omega\mathbf{S}^-\cdot\mathbf{x}) \tag{5.5}$$

and, from (3.4), the mean energy flux vector is

$$\tilde{\mathbf{R}} = \tfrac{1}{2}\omega^2 S^+\{(\lambda^+ + \mu^+)(\mathbf{U}^+\cdot\mathbf{m})\mathbf{U}^+ + \mu^+(\mathbf{U}^+\cdot\mathbf{U}^+)\mathbf{m}\}\exp(-2\epsilon\omega\mathbf{S}^-\cdot\mathbf{x}). \tag{5.6}$$

Eq. (5.4) yields two different solutions for $(S^+)^2$ and it is seen from $(5.3)_1$ that these correspond to waves for which the vector $\mathbf{U}^+$ is polarized transversely and longitudinally with respect to $\mathbf{m}$, the unit normal to the planes of constant phase. We shall discuss the two types of wave separately.

(i) *Transverse waves.* In this case, we have

$$\mu^+(S^+)^2 = \rho, \quad \mathbf{U}^+\cdot\mathbf{m} = 0. \tag{5.7}$$

Eq. $(5.3)_2$ then becomes

$$S^-[2\mu^+(\mathbf{m}\cdot\mathbf{n})\mathbf{U}^+ + (\mu^+ + \lambda^+)(\mathbf{U}^+\cdot\mathbf{n})\mathbf{m}] + S^+\mu^-\mathbf{U}^+ = -S^+(\mu^+ + \lambda^+)(\mathbf{U}^-\cdot\mathbf{m})\mathbf{m}, \tag{5.8}$$

and (5.5) and (5.6) become

$$\begin{aligned} \tilde{\mathfrak{D}} &= -\epsilon\pi\omega^2(S^+)^2\mu^-\mathbf{U}^+\cdot\mathbf{U}^+ \exp(-2\epsilon\omega\mathbf{S}^-\cdot\mathbf{x}), \\ \tilde{\mathfrak{R}} &= \tfrac{1}{2}\omega^2S^+\mu^+(\mathbf{U}^+\cdot\mathbf{U}^+)\mathbf{m} \exp(-2\epsilon\omega\mathbf{S}^-\cdot\mathbf{x}). \end{aligned} \tag{5.9}$$

Taking the inner product of (5.8) with $\mathbf{U}^+$ and using $(5.7)_2$, we obtain

$$2\mu^+S^-(\mathbf{m}\cdot\mathbf{n}) = -\mu^-S^+. \tag{5.10}$$

Taking the inner product of (5.8) with $\mathbf{m}$ and using $(5.7)_2$, we obtain

$$S^+\mathbf{U}^-\cdot\mathbf{m} = -S^-\mathbf{U}^+\cdot\mathbf{n}. \tag{5.11}$$

Thus, provided

$$\mathbf{m}\cdot\mathbf{n} \neq 0, \tag{5.12}$$

Eq. (5.10) determines S^- and Eq. (5.11) determines the component of $\mathbf{U}^-$ parallel to $\mathbf{m}$, i.e. normal to the planes of constant phase. The component of $\mathbf{U}^-$ perpendicular to $\mathbf{m}$ is, however, not determined by the equations.

If $\mathbf{m}\cdot\mathbf{n} = 0$, so that the condition (5.12) is violated, then, bearing in mind that $S^+ \neq 0$, i.e. the wave cannot have zero phase velocity, we must have $\mu^- = 0$. This condition implies that the material is elastic with respect to shear vibrations. Conversely, if $\mu^- = 0$, then $S^- = 0$, unless $\mathbf{m}\cdot\mathbf{n} = 0$ (i.e., the planes of constant phase and constant amplitude are perpendicular). The latter condition arises in the case of Love and Rayleigh waves. If $S^- = 0$, the planes of constant phase are also planes of constant amplitude and the wave propagates without change of amplitude. Also, from (5.11) it follows that $\mathbf{U}^-\cdot\mathbf{m} = 0$, i.e. the wave is entirely transverse. We therefore conclude that for an elastic material, the waves must either have their planes of constant phase parallel or perpendicular to planes of constant amplitude. In the latter case, i.e., if $\mu^- = 0$ and $\mathbf{m}\cdot\mathbf{n} = 0$, then Eq. (5.10) is identically satisfied and does not enable us to determine S^-. The ratio $S^-/(\mathbf{U}^-\cdot\mathbf{m})$ is determined from (5.11). S^- and the component of $\mathbf{U}^-$ in the direction of $\mathbf{U}^+$ must be determined from the boundary conditions.

In the case when $\mathbf{m}\cdot\mathbf{n} \neq 0$ and $\mu^- \neq 0$, i.e. the material is not elastic, it follows from (5.10) that $S^- \neq 0$. Then, from (5.10) and (5.11) it follows that

$$\mathbf{U}^-\cdot\mathbf{m} = \frac{1}{2}\frac{\mu^-}{\mu^+}\frac{\mathbf{U}^+\cdot\mathbf{n}}{\mathbf{m}\cdot\mathbf{n}}. \tag{5.13}$$

The waves will be elliptically polarized, the plane of polarization being, in general, inclined at a small angle to the planes of constant phase. It follows from (5.13) that if $\mathbf{m} = \mathbf{n}$, so that $\mathbf{U}^+\cdot\mathbf{n} = 0$, then $\mathbf{U}^-\cdot\mathbf{m} = 0$ and the plane of polarization is perpendicular to the direction of propagation $\mathbf{m}$. We note also that if $\mu^- \neq 0$, the planes of constant amplitude and constant phase may be inclined to each other at any angle except a right angle. However, as far as the present discussion is concerned, this is restricted by the consideration that $\mathbf{m}\cdot\mathbf{n}$ must not be so small that the condition that the magnitude of $\epsilon\mathbf{U}^-$ be small compared with that of $\mathbf{U}^+$ is violated.

(ii) *Longitudinal waves.* In this case we have

$$(\lambda^+ + 2\mu^+)(S^+)^2 = \rho, \quad \mathbf{U}^+ = \beta\mathbf{m}, \tag{5.14}$$

say, where β is a real constant. Then $(5.3)_2$ becomes

$$\beta S^+ S^-[2\mu^+(\mathbf{m}\cdot\mathbf{n})\mathbf{m} + (\lambda^+ + \mu^+)\{\mathbf{n} + (\mathbf{m}\cdot\mathbf{n})\mathbf{m}\}] + \beta(S^+)^2(\lambda^- + 2\mu^-)\mathbf{m}$$
$$= [\rho - \mu^+(S^+)^2]\mathbf{U}^- - (\lambda^+ + \mu^+)(S^+)^2(\mathbf{U}^-\cdot\mathbf{m})\mathbf{m}, \tag{5.15}$$

and (5.5) and (5.6) become

$$\begin{aligned}\bar{\mathfrak{D}} &= -\epsilon\pi\beta^2\omega^2(S^+)^2(\lambda^- + 2\mu^-)\exp(-2\epsilon\omega\mathbf{S}^-\cdot\mathbf{x}),\\ \bar{\mathfrak{R}} &= \tfrac{1}{2}\beta^2\omega^2 S^+(\lambda^+ + 2\mu^+)\mathbf{m}\exp(-2\epsilon\omega\mathbf{S}^-\cdot\mathbf{x}).\end{aligned} \tag{5.16}$$

Multiplying (5.15) throughout by $\mathbf{m}$, we obtain, with $(5.14)_1$

$$2S^-(\lambda^+ + 2\mu^+)\mathbf{m}\cdot\mathbf{n} = -S^+(\lambda^- + 2\mu^-). \tag{5.17}$$

Using $(5.14)_1$ and assuming $S^+ \neq 0$, we can rewrite (5.15) as

$$\beta S^-\{(\lambda^+ + 3\mu^+)(\mathbf{m}\cdot\mathbf{n})\mathbf{m} + (\lambda^+ + \mu^+)\mathbf{n}\} + \beta S^+(\lambda^- + 2\mu^-)\mathbf{m}$$
$$= S^+(\lambda^+ + \mu^+)[\mathbf{U}^- - (\mathbf{U}^-\cdot\mathbf{m})\mathbf{m}]. \tag{5.18}$$

We note that $\mathbf{U}^- - (\mathbf{U}^-\cdot\mathbf{m})\mathbf{m}$ is the component of $\mathbf{U}^-$ perpendicular to $\mathbf{m}$, i.e. to $\mathbf{U}^+$. Thus, provided

$$\mathbf{m}\cdot\mathbf{n} \neq 0, \tag{5.19}$$

Eq. (5.17) determines S^- and Eq. (5.18) enables us to determine the component of $\mathbf{U}^-$ perpendicular to $\mathbf{m}$, its component parallel to $\mathbf{m}$ (and hence the phase of $\mathbf{U}\cdot\mathbf{m}$) remaining undetermined.

We note also from (5.18) that $[\mathbf{U}^- - (\mathbf{U}^-\cdot\mathbf{m})\mathbf{m}]$ lies in the plane of $\mathbf{m}$ and $\mathbf{n}$ and accordingly the wave is, in general, elliptically polarized in the plane of $\mathbf{m}$ and $\mathbf{n}$. However, in the particular case when $\mathbf{n} = \mathbf{m}$, i.e., the planes of constant phase and constant amplitude are parallel, it follows from (5.17) and (5.18) that $\mathbf{U}^- - (\mathbf{U}^-\cdot\mathbf{m})\mathbf{m} = 0$, so that the wave is linearly polarized in the direction of $\mathbf{m}$.

If $\mathbf{m}\cdot\mathbf{n} = 0$, so that the condition (5.19) is violated, it follows from (5.17) that $\lambda^- + 2\mu^- = 0$, i.e., the material is elastic with respect to longitudinal waves. Conversely, if the material is elastic so that $\lambda^- + 2\mu^- = 0$, then either $\mathbf{m}\cdot\mathbf{n} = 0$, or $S^- = 0$. In the former case, the planes of constant phase and constant amplitude are perpendicular and we note that (5.17) is identically satisfied. Accordingly, Eqs. (5.17) and (5.18) cannot be used to determine S^- and $\mathbf{U}^-$. These must then be determined from the boundary conditions. If $S^- = 0$, it follows from (5.18) that $[\mathbf{U}^- - (\mathbf{U}^-\cdot\mathbf{m})\mathbf{m}] = 0$, i.e., the wave is longitudinally polarized.

In the case when $\lambda^- + 2\mu^- \neq 0$, i.e., the material is not elastic, it follows from (5.17) that $S^- \neq 0$. Then, from (5.17) and (5.18), we obtain

$$\mathbf{U}^- - (\mathbf{U}^- \cdot \mathbf{m})\mathbf{m} = \frac{\alpha(\lambda^- + 2\mu^-)}{2(\lambda^+ + 2\mu^+)\mathbf{m} \cdot \mathbf{n}} [(\mathbf{m} \cdot \mathbf{n})\mathbf{m} - \mathbf{n}]. \tag{5.20}$$

Accordingly, as in the case of transverse waves, the planes of constant phase and constant amplitude may be inclined to each other at any angle except a right angle. However, as far as the present discussion is concerned, this is restricted by the consideration that $\mathbf{m} \cdot \mathbf{n}$ must not be so small that the condition that the magnitude of $\epsilon \mathbf{U}^-$ be small compared with that of $\mathbf{U}^+$ is violated.

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