ON A UNIQUE NONLINEAR OSCILLATOR*

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In this note we present a remarkable nonlinear system which has the property that all its bounded periodic motions are simple harmonic. The system is a particle obeying the highly nonlinear equation of motion

$$\ddot{x} + \frac{(\alpha - \lambda \dot{x}^2)}{(1 + \lambda x^2)} x = 0. \tag{1}$$

This equation is obtainable from the Lagrangian

$$L = \tfrac{1}{2}[(\dot{x}^2 - \alpha x^2)/(1 + \lambda x^2)], \tag{2}$$

whence the canonical momentum p and the Hamiltonian H may be obtained:

$$p = \partial L/\partial \dot{x} = \dot{x}(1 + \lambda x^2)^{-1}, \tag{3}$$

$$H = \tfrac{1}{2}[p^2(1 + \lambda x^2) + \alpha x^2(1 + \lambda x^2)^{-1}]. \tag{4}$$

The Lagrangian (2) is the single particle analogue of the Lagrangian density

$$L = \tfrac{1}{2}[\{(\partial_\rho \varphi)^2 - m^2 \varphi^2\}/(1 + \lambda \varphi^2)] \qquad (\rho = 0, 1, 2, 3) \tag{5}$$

of a relativistic scalar field. Field systems with this kind of Lagrangian are of wide current interest [1, 2] in the context of elementary particle theory, and have in fact been considered especially in the context of chiral Lagrangian theories of pion interactions. Here we limit ourselves to obtaining the solutions of Eq. (1), which immediately lead to the plane wave (c-number) solutions of the field equation of (5) also.

It is known [3] that the first integral of the general nonlinear differential equation

$$y'' + f(y)y'^2 + g(y) = 0 \tag{6}$$

can be expressed as

$$(y')^2 \exp [2\xi(y)] = -(\epsilon/\lambda) - 2 \int g(y) \exp [2\xi(y)]\, dy, \tag{7}$$

where

$$\xi(y) = \int f(y)\, dy$$

and the constant of integration has been written as $-(\epsilon/\lambda)$ for future convenience.

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In the case of our Eq. (1), this solution reduces to

$$(\dot{x})^2 + \epsilon x^2 = \lambda^{-1}(-\epsilon + \alpha). \tag{8}$$

Considering first the case $\lambda > 0$, we note that when ϵ is such that $\alpha > \epsilon > 0$, the phase trajectories ($x - \dot{x}$ curves) given by (8) are ellipses which can be conveniently represented as

$$(\dot{x})^2 + \omega^2 x^2 = \omega^2 A^2, \tag{9}$$

where

$$A^2 = \frac{1}{\lambda}\left(-1 + \frac{\alpha}{\epsilon}\right), \qquad \omega = \epsilon = \frac{\alpha}{1 + \lambda A^2}. \tag{10}$$

Solutions of (9) have the *simple harmonic form*

$$x = A \sin(\omega t + \theta). \tag{11}$$

The energy associated with motion is obtained from (4) as

$$E = \frac{1}{2}\omega^2 A^2 = \frac{1}{2}\frac{\alpha A^2}{(1 + \lambda A^2)}. \tag{12}$$

The nonlinear nature of the original equation manifests itself through the dependence of various quantities (including ω) on A. From (10), (11) and (12) we find that as $A \rightarrow \infty$, $\omega \rightarrow 0$, $\dot{x}$ tends to the constant finite value $(\alpha/\lambda)^{1/2}$ and the energy remains finite, $E \rightarrow (\alpha/2\lambda)$. The role of the value $(\alpha/\lambda)^{1/2}$ of $\dot{x}$ as a "bifurcation point" could be seen directly from the fact that the force term in the equation of motion (1) changes from attractive to repulsive (for all x) as $\dot{x}$ increases beyond this value. Solutions of (1) appropriate to this regime ($\dot{x} > (\alpha/\lambda)^{1/2}$) correspond to $\epsilon < 0$ in (8), when the phase trajectories become hyperbolas, and one has

$$x = B \sinh(\mu t + \theta) \tag{13a}$$

$$B^2 = \frac{1}{\lambda}\left(\frac{\alpha}{|\epsilon|} + 1\right), \qquad \zeta^2 = |\epsilon| = \frac{-\alpha}{1 - \lambda B^2}. \tag{13b}$$

The phase trajectories for various values of ϵ (both positive and negative) are shown in Fig. 1a. For $\epsilon = 0$, the trajectory consists of either of the straight lines $\dot{x} = \pm(\alpha/\lambda)^{1/2}$.

If $\lambda < 0$, once again there are elliptical and hyperbolical phase trajectories, but the bifurcation point now is not in $\dot{x}$ but in x, at $x = \pm |\lambda|^{-1/2}$. This is reflected in the existence of an upper limit A max $= |\lambda|^{-1/2}$ on the amplitude of the simple harmonic oscillations $x = A \sin(\omega t + \theta)$ and as this maximum is approached, the energy E which is now given by $E = \frac{1}{2}\alpha^2 A^2/(1 - |\lambda| A^2)$ goes to infinity. These facts will be clear from the phase trajectories given in Fig. 1b. The stability of the periodic solutions are under investigation and will be reported separately.

It is interesting to compare the equation

$$\ddot{x} + \alpha\frac{x}{1 + \lambda x^2} + \lambda\frac{x}{(1 + \lambda x^2)}(\dot{x})^2 = 0 \tag{14}$$

with Eq. (1), the difference between the two being only in the sign of the last term. Eq. (14) has been discussed in the literature [4, 5] in connection with the motion of a material point along a parabola rotating about its vertical axis with constant angular

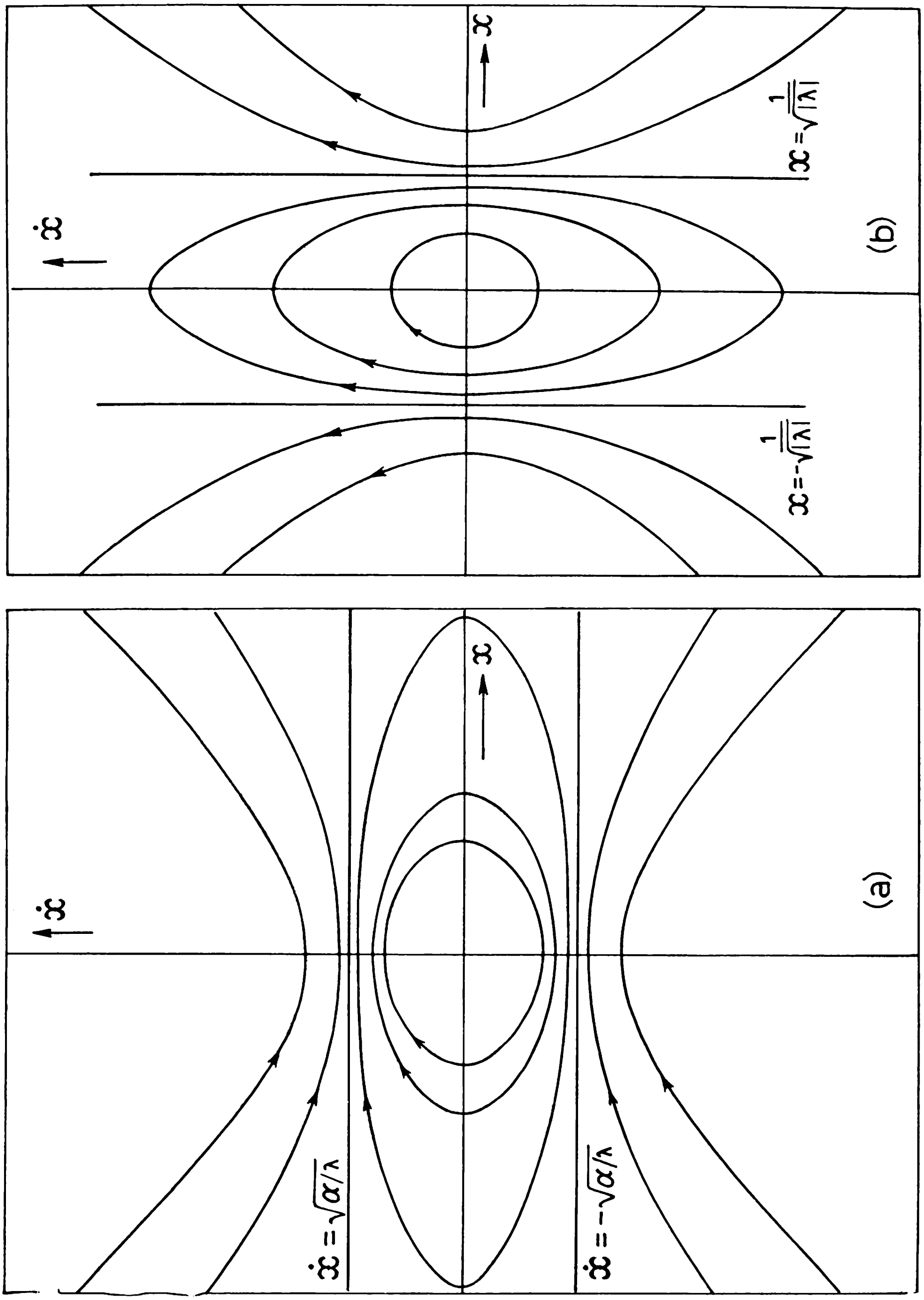

FIGURE I

FIG. 1. The phase trajectories ($x - \dot{x}$ curves) of the nonlinear oscillator: (*a*) $\lambda > 0$, (*b*) $\lambda < 0$.

velocity. The integral curves of this equation are described by the relation

$$\tfrac{1}{2}[(1 + \lambda x^2)\dot{x}^2 + \alpha x^2] = E = \text{const.} \tag{15}$$

The analytic solutions of Eq. (15) involve complicated elliptic integrals of the second kind and so are difficult to deal with.

Finally it is pertinent to point out that the Euler–Lagrange equation of motion for the Lagrangian (5)

$$(1 + \lambda\varphi^2) \Box \varphi + m^2\varphi - \lambda\varphi(\partial_\rho\varphi)^2 = 0 \tag{16}$$

($\Box = \partial_\rho\partial_\rho$) admits Lorentz invariant plane wave solutions of the form $\varphi(\mathbf{x}, t) = f(\mathbf{k}\cdot\mathbf{x} - \omega t)$. In terms of the variable $\zeta = (\mathbf{k}\cdot\mathbf{x} - \omega t)$ Eq. (16) becomes

$$\frac{d^2f}{d\zeta^2} + \frac{m^2}{\omega^2 - k^2}\frac{f}{(1 + \lambda f^2)} - \frac{\lambda f}{(1 + \lambda f^2)}\left(\frac{df}{d\zeta}\right)^2 = 0. \tag{17}$$

This is similar to our earlier Eq. (1). When $(\omega^2 - k^2) > 0$, real nonsingular solutions exist for either sign of λ which can be given as

$$\varphi(\mathbf{x}, t) = A \sin (\mathbf{k}\cdot\mathbf{x} - \omega t + \theta) \tag{18}$$

with

$$(\omega^2 - k^2) = m^2/(1 + \lambda A^2). \tag{19}$$

For $\lambda < 0$, this is valid only in the region $0 \leq |A| \leq (1/\sqrt{|\lambda|})$. Tachyon-like solutions having the form (18) but with $(\omega^2 - k^2) < 0$ also exist provided $\lambda < 0$. In this case

$$(\omega^2 - k^2) = -m^2/(|\lambda| A^2 - 1) \tag{20}$$

with $|\lambda| A^2$ necessarily exceeding unity. We also do have complex solutions of the form

$$\varphi = A \exp [i(\mathbf{k}\cdot\mathbf{x} - \omega t + \theta)] \tag{21}$$

where $(\omega^2 - k^2) = m^2$. Such solutions are of immense interest [6] in connection with the search of nonperturbative solutions of the corresponding quantum field equation [7]. Quantization of the Eqs. (2) and (5) will be dealt with elsewhere.

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