QUARTERLY OF APPLIED MATHEMATICS

SOME MATHEMATICAL MODELS FOR POPULATION DYNAMICS THAT LEAD TO SEGREGATION*

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I. Introduction.   In this paper I develop several mathematical models for population flow within a region containing two or more distinct cultural groups. My main purpose is to pinpoint mathematically those mechanisms which give rise to segregation. For this reason the models presented are quite simple, both from a sociological and from a mathematical viewpoint.

The first model is the simplest: people are allowed to flow into and out of the region, and population change due to birth and death is included; population flow within the region, however, is not considered. For two groups that repel each other I establish the existence of a number A, called the *tipping ratio*,[1] with the following property: if $\mathring{n}_1$ and $\mathring{n}_2$ are the initial populations of the two groups, then

(i) when $\mathring{n}_1/\mathring{n}_2 < A$, the population of group 1 tends to zero in a finite time;

(ii) when $\mathring{n}_1/\mathring{n}_2 > A$, the population of group 2 tends to zero in a finite time;

(iii) when $\mathring{n}_1/\mathring{n}_2 = A$, both populations remain positive for all time.

In the second model I allow for population flow within the region. At a given point x this flow is governed by the population densities in an arbitrarily small neighborhood of x; in fact, the population flux at x (for each cultural group) is assumed to be a linear function of the density gradients at x. For two groups that repel each other this theory leads to the "backward heat equation" for the difference in population densities, and this, in turn, leads to the conclusion that population *differences* grow without bound.

This second model bears some similarity to classical mixture theory. The similarity, however, ends with the form of the basic equations. Whereas in mixture theory the flow tends to equilibrium, in the present theory the repulsion of one group toward another leads to unstable behavior, and ultimately to separation of the species.

Models of the second type are probably appropriate for the study of areas in which population flow is local in nature. The third model I consider allows for long-range

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[1] The phenomenon of neighborhood tipping is discussed in detail by Grodzins [1] and Schelling [2]. According to Schelling [2], "tipping is said to occur when a recognizable new minority enters a neighborhood in sufficient numbers to cause the earlier residents to begin evacuating."

population movement; it should be applicable, e.g., to situations in which people move from a rural area to a distant urban area. I prove under certain simplifying assumptions that such a model again leads to segregation whenever the groups repel each other.

The theories presented here are continuum theories. (A completely different approach is taken by Schelling [2].) They confine themselves to relations among phenomenological variables such as population density, population flux, and rate of population supply, and neglect behavior on a small scale. It is hoped that the recent advances in modern continuum physics can ultimately be brought to bear on the difficult problems of population dynamics; this paper is a modest start in that direction.

II. Spatially-independent theory. 1. *Basic equations.* Consider a given region R occupied by m distinct cultural groups $\alpha = 1, \cdots, m$. At any given time t, let $n_\alpha(t)$ be the number of people of group α living in R, and let $s_\alpha(t)$ denote the number of people of group α entering (or leaving) R per unit time, e.g., by births and deaths, and by movement to and from areas outside of R. For each group the rate of population increase must equal the number of people entering the region per unit time; therefore, for each α,

$$\dot{n}_\alpha = s_\alpha , \qquad (1.1)$$

where the superposed dot denotes differentiation with respect to t.

We assume that each supply s_α is a function of the populations $n_1, \cdots, n_m$:

$$s_\alpha = f_\alpha(n_1, \cdots, n_m); \qquad (1.2)$$

thus

$$\dot{n}_\alpha = f_\alpha(n_1, \cdots, n_m). \qquad (1.3)$$

The basic problem of population dynamics, within this context, is to find $n_\alpha(t)$ for $t \geq 0$ given the population $\overset{\circ}{n}_\alpha$ of each α at the initial time $t = 0$.

In general, the functions f_α will be nonlinear. However, in order to gain at least a qualitative understanding of the subject, we assume that

$$f_\alpha(n_1, \cdots, n_m) = \sum_{\beta=1}^{m} a_{\alpha\beta} n_\beta , \qquad (1.4)$$

where the $a_{\alpha\beta}$ are constants. In (1.4) terms of the form $a_{\alpha\alpha} n_\alpha$ represent the rate of increase of α due only to the presence of α, e.g., by births and deaths. On the other hand, $a_{\alpha\beta} n_\beta$ ($\alpha \neq \beta$) represents the rate of increase of α due to the presence of β. For situations in which the groups α and β *repel* each other, the constant $a_{\alpha\beta}$ will be negative.

In view of (1.4), the dynamical system (1.3) takes the form

$$\dot{n}_\alpha = \sum_{\beta=1}^{m} a_{\alpha\beta} n_\beta . \qquad (1.5)$$

In the next section we will study solutions of (1.5) for the special case of two cultural groups.

2. *Two cultural groups.* For two cultural groups (1.5) takes the form[2]

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[2] Models such as this have a long history. Keyfitz ([3], pp. 271–284) discusses physical situations described by the system (2.1). In particular, on p. 278 he gives a detailed description of the phase trajectories for the case in which $a_{11} > 0$, $a_{22} > 0$, and (2.2) hold (his *competition* model).

$$\dot{n}_1 = a_{11}n_1 + a_{12}n_2, \qquad \dot{n}_2 = a_{21}n_1 + a_{22}n_2. \tag{2.1}$$

We wish to study situations in which the two groups repel each other; thus we assume that

$$a_{12} < 0, \qquad a_{21} < 0. \tag{2.2}$$

With this assumption the system (2.1) has the solution

$$n_1(t) = e^{\lambda t}\left\{\frac{1}{\omega}(a_{12}\mathring{n}_2 + \beta\mathring{n}_1)\sinh\omega t + \mathring{n}_1\cosh\omega t\right\},$$

$$n_2(t) = e^{\lambda t}\left\{\frac{1}{\omega}(a_{21}\mathring{n}_1 - \beta\mathring{n}_2)\sinh\omega t + \mathring{n}_2\cosh\omega t\right\}, \tag{2.3}$$

where

$$\beta = \tfrac{1}{2}(a_{11} - a_{22}), \quad \lambda = \tfrac{1}{2}(a_{11} + a_{22}), \quad \omega = (\beta^2 + a_{12}a_{21})^{1/2}, \quad \mathring{n}_1 = n_1(0), \quad \mathring{n}_2 = n_2(0).$$

A careful study of (2.3) leads to the conclusion that $n_1(t) \to 0$ in a finite time if and only if $\mathring{n}_1/\mathring{n}_2 < A$; $n_2(t) \to 0$ in a finite time if and only if $\mathring{n}_1/\mathring{n}_2 > A$, where

$$A = \frac{\omega - \beta}{-a_{21}} = \frac{-a_{12}}{\omega + \beta}. \tag{2.4}$$

Thus we have the following result: if the differential equations (2.1), subject to (2.2), model the behavior of two groups living in a region R, then R *will become segregated in a finite time* for all choices of the initial populations except those satisfying

$$\mathring{n}_1/\mathring{n}_2 = A. \tag{2.5}$$

We call A the *tipping ratio*: If $\mathring{n}_1/\mathring{n}_2 < A$ the region will eventually be populated only by group 2; if $\mathring{n}_1/\mathring{n}_2 > A$ the region will eventually contain only group 1.

If (2.5) holds, then $n_1(t) > 0$ and $n_2(t) > 0$ for all time t. Further, when $\lambda < \omega$ (respectively, $\lambda > \omega$) both $n_1(t)$ and $n_2(t)$ tend to zero (respectively, infinity) as $t \to \infty$; when $\lambda = \omega$, n_1 and n_2 remain constant for all time.

The time T at which, say, $n_1(T) = 0$[3] (when $\mathring{n}_1/\mathring{n}_2 < A$) is called the *segregation time* and, by $(2.3)_1$, is given by

$$T = \frac{1}{\omega}\tanh^{-1}\left(\frac{\omega}{\alpha\omega + (\alpha - 1)\beta}\right), \qquad \alpha = A\mathring{n}_2/\mathring{n}_1 > 1.$$

For ω and β fixed, T is infinite when $\alpha = 1$ and *decreases* monotonically to zero as α increases to infinity. Thus the closer $\mathring{n}_1/\mathring{n}_2$ is to the tipping ratio A, the longer it will take for n_1 to vanish.

It is clear from the preceding analysis that to avoid segregation the initial populations should have a ratio equal to A. When this condition is not satisfied, the closer this ratio is to A, the longer it will take for segregation to occur. The implications of this result are easier to interpret in situations for which

$$\epsilon = \frac{\beta^2}{a_{12}a_{21}} = \frac{(a_{11} - a_{22})^2}{4a_{12}a_{21}}$$

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[3] The theory is not valid for $t > T$, since for those times it generally predicts negative values for n_1.

is small; i.e., situations in which population movement due to group repulsion is large compared to population increases due to natural causes. In this instance

$$A = (a_{12}/a_{21})^{1/2} + O(\epsilon),$$

$$T = \frac{1}{(a_{12}a_{21})^{1/2}} \tanh^{-1}\left[\frac{\overset{\circ}{n_1}}{\overset{\circ}{n_2}}\left(\frac{a_{21}}{a_{12}}\right)^{1/2}\right] + O(\epsilon) \qquad (2.6)$$

as $\epsilon \to 0$, so that segregation will be prolonged as long as $(\overset{\circ}{n_1}/\overset{\circ}{n_2})^2$ is close to a_{12}/a_{21}. It is interesting to note that if $a_{11} = a_{22}$, then $\beta = 0$ and (2.6) holds with the $0(\epsilon)$ terms zero, so that the tipping ratio A and the segregation time T are independent of $a_{11} = a_{22}$.

Our analysis can be applied, at least roughly, to interactions between blacks and whites. Let $b = \overset{\circ}{n_1}$ and $w = \overset{\circ}{n_2}$ denote, respectively, the initial black and white populations of a neighborhood, and let $a_{12} = a_{bw}$ = the rate at which whites repel blacks, $a_{21} = a_{wb}$ = the rate at which blacks repel whites. A study by Duncan and Duncan [4] found no instance between 1940 and 1950 of a mixed neighborhood (25%—75% white) in which succession from white to black was arrested or reversed. Grodzins [2] estimates that whites begin to evacuate a neighborhood when it becomes 20% black. Using Grodzins' figure to estimate the tipping ratio, we find that $A = \frac{1}{4}$, and that $b/w > \frac{1}{4}$ results in an all-black neighborhood. Taking the estimate (2.6) for A as exact, we see that $a_{wb}/a_{bw} = 16$, so that the rate at which blacks repel whites is sixteen times the rate at which whites repel blacks.

A case of interest not included in (2.2) is that for which, say, $a_{12} = 0$. In this instance the population of group 1 will increase exponentially, but the population of group 2, which is repelled by group 1, will eventually vanish.

III. Theory with local population transport. 3. *Basic equations.* We now generalize our theory to allow for population movement in one space dimension. Thus the underlying flow region is assumed to be a closed interval $[0, L]$ of the real line. At any given point x in $[0, L]$ and time t, let $n_\alpha(x, t)$ denote the *density* of group α, let $h_\alpha(x, t)$ denote the *population flux* of α, and let $s_\alpha(x, t)$ denote the *population supply* of α. The density $n_\alpha(x, t)$ is the number of people, per unit length, of group α at x. The population flux $h_\alpha(x, t)$ is the number of people of group α that pass the point x per unit time; $h_\alpha > 0$ (respectively, $h_\alpha < 0$) signifies that the flow is in the direction of increasing (respectively, decreasing) x. The population supply $s_\alpha(x, t)$ represents the number of people, per unit length and time, of group α supplied directly at x, e.g., by births and deaths, and by movement from areas outside of $[0, L]$. We postulate the following *law of population balance*:

$$\frac{d}{dt}\int_a^b n_\alpha(x, t)\, dx = h_\alpha(a, t) - h_\alpha(b, t) + \int_a^b s_\alpha(x, t)\, dx. \qquad (3.1)$$

In words, (3.1) asserts that the rate at which the population of α in $[a, b]$ is changing is equal to the number of people of α that enter $[a, b]$ per unit time. Eq. (3.1) is assumed to hold for every such interval $[a, b]$, for every group α, and for every time t. If the fields n_α, h_α, and s_α are sufficiently well behaved, then (1) is equivalent to the following partial differential equation:

$$\dot{n}_\alpha = -(\partial h_\alpha/\partial x) + s_\alpha, \qquad (3.2)$$

where the superposed dot denotes partial differentiation with respect to time t holding the point x fixed.

We now assume that the population flux at x is a function of the density variation in the immediate vicinity of x. The simplest hypothesis consistent with this assumption is that for each group α the population flux h_α is a linear function of the density gradients $\partial n_1/\partial x$, $\partial n_2/\partial x$, $\cdots$, $\partial n_m/\partial x$. Thus we assume that

$$h_\alpha = -\sum_{\beta=1}^{m} c_{\alpha\beta} \frac{\partial n_\beta}{\partial x}, \qquad (3.3)^4$$

where the $c_{\alpha\beta}$ are constants, and where the negative sign is chosen for convenience. The term $-c_{\alpha\beta}\,\partial n_\beta/\partial x$ gives the population flux of group α due to the presence of β. If, e.g., $c_{\alpha\beta}$ is positive, then this term leads to a flow of group α in the direction of decreasing β; thus $c_{\alpha\beta} > 0$ $(\alpha \neq \beta)$ models situations in which group β repels group α.

We assume that the population flux s_α is given by (1.2) and (1.4); therefore by (3.2) and (3.3) we have the following system of partial differential equations:

$$\dot{n}_\alpha = \sum_{\beta=1}^{m} c_{\alpha\beta} \frac{\partial^2 n_\beta}{\partial x^2} + \sum_{\beta=1}^{m} a_{\alpha\beta} n_\beta . \qquad (3.4)$$

The boundary-initial-value problem associated with this equation can be stated as follows: given the flow region $[0, L]$, the matrices $||a_{\alpha\beta}||$ and $||c_{\alpha\beta}||$, an initial density distribution $n_\alpha(x)$ (for each α), and the population flux $\bar{h}_\alpha(t)$ and $\hbar_\alpha(t)$ at the boundary points 0 and L (for each α); find a solution $n_\alpha(x, t)$ $(\alpha = 1, 2, \cdots, m)$ of (3.4) for $0 < x < L$ and $t > 0$ that satisfies the initial conditions

$$n_\alpha(x, 0) = \mathring{n}_\alpha(x) \qquad (0 < x < L) \qquad (3.5)$$

and the boundary conditions

$$h_\alpha(0, t) = \bar{h}_\alpha(t), \qquad h_\alpha(L, t) = \hbar_\alpha(t) \qquad (t > 0), \qquad (3.6)$$

with h_α given by (3.3).

For the case in which the transport matrix $||c_{\alpha\beta}||$ is symmetric and positive definite, the system (3.4) is parabolic, and the above problem is well-posed (i.e., the solution exists, is unique, and depends continuously on the data). Unfortunately, however, for most problems of interest $||c_{\alpha\beta}||$ will *not* have these properties.

The above theory is easily generalized to two space dimensions. Indeed, in this case h_α is a vector in R^2, and (3.2) and (3.3) have the forms:

$$\dot{n}_\alpha = -\mathrm{div}\, h_\alpha + s_\alpha , \qquad (3.7)$$

$$h_\alpha = -\sum_{\beta=1}^{m} c_{\alpha\beta} \nabla n_\beta , \qquad (3.8)$$

where div and ∇ are, respectively, the divergence and gradient operators in R^2. In writing (3.8) the assumption of isotropy is tacit. If the population flow is anisotropic (different in different directions), then (3.8) still holds, but with the $c_{\alpha\beta}$ interpreted as linear transformations from R^2 to R^2.

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4 Constitutive equations of this type were first proposed by Kerner [5], who gives a detailed physical argument for their validity, at least in studies concerning populations of animals, plants, bacteria, etc. In this connection see also Skellam [6].

4. *Two cultural groups.* In this instance (3.3) reduces to

$$h_1 = -c_{11} \frac{\partial n_1}{\partial x} - c_{12} \frac{\partial n_2}{\partial x} \qquad h_2 = -c_{21} \frac{\partial n_1}{\partial x} - c_{22} \frac{\partial n_2}{\partial x}. \tag{4.1}$$

For convenience, we restrict our attention to situations in which:

(i) the population flux of each group depends only on the density gradient of the other group, i.e.

$$c_{11} = c_{22} = 0; \tag{4.2}$$

(ii) the population flux of each group always lies in the direction of *decreasing* density of the other group, i.e.

$$c_{12} > 0, \qquad c_{21} > 0. \tag{4.3}$$

In addition, we suppose, for convenience, that

$$c_{12} = c_{21} = c, \qquad ||a_{\alpha\beta}|| = 0. \tag{4.4}$$

Then (3.4) takes the form

$$\dot{n}_1 = c(\partial^2 n_2/\partial x^2), \qquad \dot{n}_2 = c(\partial^2 n_1/\partial x^2). \tag{4.5}$$

We assume further that the flow region is *isolated* in the sense that $\bar{h}_\alpha = \tilde{h}_\alpha = 0$ ($\alpha = 1, 2$), or equivalently that

$$\partial n_1/\partial x = \partial n_2/\partial x = 0 \qquad (x = 0, L; t > 0). \tag{4.6}$$

Since, by $(4.4)_3$, $s_\alpha = 0$, this condition implies that the *total* population of each group remains constant in time.

If we define the total density n and the density difference u through the relations $n = n_1 + n_2$, $u = n_1 - n_2$, then the problem (4.5), (4.6), and (3.5) is equivalent to the following pair of problems:

$$\dot{n} = c(\partial^2 n/\partial x^2),$$
$$\partial n/\partial x = 0 \qquad\qquad (x = 0, L; t > 0), \tag{4.7}$$
$$n(x, 0) = \overset{\circ}{n}(x) \qquad\qquad (0 < x < L).$$

$$\dot{u} = -c(\partial^2 u/\partial x^2),$$
$$\partial u/\partial x = 0 \qquad\qquad (x = 0, L; t > 0), \tag{4.8}$$
$$u(x, 0) = \overset{\circ}{u}(x) \qquad\qquad (0 < x < L).$$

As is well known, the problem (4.7) for the total density n is "well-posed". Further, the solution is stable, and *the total density n tends to a constant C*:

$$n(x, t) \to C \quad \text{as} \quad t \to \infty, \tag{4.9}$$

with

$$C = \frac{1}{L} \int_0^L \overset{\circ}{n}(x) \, dx. \tag{4.10}$$

On the other hand, the density difference u obeys the "backward heat equation" $(4.8)_1$; consequently, the problem (4.8) is not well-posed (see, e.g., Friedman [7], pp.

172–178). Indeed, this problem is quite unstable: any initial fluctuation in the density distribution will grow without bound. For example, consider the initial data

$$\mathring{u}(x) = \epsilon \cos(k\pi x/L) \qquad (4.11)$$

with k an odd integer. A solution of (4.8), (4.11) is

$$u(x, t) = \mathring{u}(x)e^{\lambda t}, \qquad \lambda = \frac{ck^2\pi^2}{L^2} > 0. \qquad (4.12)$$

Thus no matter how small ϵ is, $|u(x, t)| \to \infty$ as $t \to \infty$ for every x with $\mathring{u}(x) \neq 0$. Moreover, the sharper the initial fluctuation, i.e. the larger the value of k, the faster the approach to infinity. Thus *if (4.2) and (4.3) hold, then differences in density grow without bound*. By (4.9) this means that for every x with $\mathring{u}(x) \neq 0$ either n_1 or n_2 will eventually vanish. Of course, the theory becomes invalid after the first such vanishing, since for larger times it predicts negative densities. Further, neither n_1 nor n_2 can vanish everywhere, since the total population of each group remains constant in time.

IV. Theory with long-range population transport. 5. *Basic equations; a simple solution.* In the model discussed in the preceding two sections the flow of population *within* the region under study is described by the flux h_α ; at a given point x, h_α is influenced only by the population density in an arbitrarily small neighborhood of x. We will now discuss a model in which people are allowed to flow between any two points, instantaneously.

Let $G_\alpha(x, y, t)$ denote the number of people of group α, per unit lengths at x and y, and per unit time, flowing from y to x at time t. Then

$$G_\alpha(x, y, t) = -G_\alpha(y, x, t), \qquad (5.1)$$

since the number of people entering x from y must equal the number of people leaving y for x. The total number of people of group α, per unit length and time, entering x from all other points in $[0, L]$ is obtained by integrating $G_\alpha(x, y, t)$ over all y in $[0, L]$:

$$g_\alpha(x, t) = \int_0^L G_\alpha(x, y, t) \, dy; \qquad (5.2)$$

then by (5.1), as would be expected,

$$\int_0^L g_\alpha(x, t) \, dx = 0. \qquad (5.3)$$

The total supply $s_\alpha(x, t)$ of group α at x is assumed to be the sum of two terms:

$$s_\alpha = g_\alpha + f_\alpha, \qquad (5.4)$$

where $f_\alpha(x, t)$ is the number of people of group α, per unit length and time, supplied directly at x by births, deaths, and movement from areas outside of $[0, L]$. As before, we assume that

$$f_\alpha = \sum_{\beta=1}^N a_{\alpha\beta}n_\beta. \qquad (5.5)$$

For convenience, let us assume that g_α is the only mechanism for population transport within the region; then $h_\alpha = 0$, and (3.2) and (5.4) yield

$$\dot{n}_\alpha = g_\alpha + f_\alpha . \tag{5.6}$$

Let

$$N_\alpha(t) = \int_0^L n_\alpha(x, t) \, dx; \tag{5.7}$$

then N_α is the total population of α. By (5.3), (5.5), and (5.6),

$$\dot{N}_\alpha = \sum_{\beta=1}^m a_{\alpha\beta} N_\beta , \tag{5.8}$$

so that $N_1 , \cdots , N_m$ obey the system (1.5) discussed at length in Sec. 1.

We now limit our discussion to two cultural groups. We assume that $G_\alpha(x, y, t)$ is proportional to the density difference $n_\beta(y, t) - n_\beta(x, t)$ $(\beta \neq \alpha)$:

$$G_\alpha(x, y, t) = \varphi_\alpha(x, y)[n_\beta(y, t) - n_\beta(x, t)] \qquad (\beta \neq \alpha). \tag{5.9}$$

The functions φ_α are assumed known *a priori*; they play a role similar to that played by the matrix $\|c_{\alpha\beta}\|$ discussed in Sec. 3. We suppose that

$$\varphi_\alpha(x, y) = \varphi_\alpha(y, x), \qquad \varphi_\alpha(x, y) \geq 0. \tag{5.10}$$

The first of these insures that (5.1) is satisfied; the second implies that people of group α flow from y to x whenever $n_\beta(y, t) > n_\beta(x, t)$ $(\beta \neq \alpha)$. Thus here, as in the last section, we restrict our attention to situations in which the two groups *repel* each other.

If we substitute (5.2) and (5.5) into (5.6), and use (5.9), we arrive at the following system of equations:

$$\dot{n}_1(x, t) = \int_0^L \varphi_1(x, y)[n_2(y, t) - n_2(x, t)] \, dy + a_{11} n_1(x, t) + a_{12} n_2(x, t),$$
$$\dot{n}_2(x, t) = \int_0^L \varphi_2(x, y)[n_1(y, t) - n_1(x, t)] \, dy + a_{21} n_1(x, t) + a_{22} n_2(x, t). \tag{5.11}$$

Let

$$k_{12}(x) = \int_0^L \varphi_1(x, y) \, dy - a_{12} , \qquad k_{21}(x) = \int_0^L \varphi_2(x, y) \, dy - a_{21} ; \tag{5.12}$$

then (5.11) takes the form

$$\dot{n}_1(x, t) = a_{11} n_1(x, t) - k_{12}(x) n_2(x, t) + \int_0^L \varphi_1(x, y) n_2(y, t) \, dy,$$
$$\dot{n}_2(x, t) = a_{22} n_2(x, t) - k_{21}(x) n_1(x, t) + \int_0^L \varphi_2(x, y) n_1(y, t) \, dy. \tag{5.13}$$

A systematic study of this system of integral-differential equations is beyond the scope of the present paper. However, if the constants $a_{\alpha\beta}$ obey the assumptions made in Sec. 2, and if $\mathring{N}_1/\mathring{N}_2$ is not equal to the tipping ratio A, then (5.8) and the results of Section 2 imply that the *total population* of one of the groups will tend to zero, i.e. *the region under consideration will become segregated*. We now consider a simple case in which the assumptions of Sec. 2 are violated. We suppose that the region is isolated; then the supplies f_α are affected only by births and deaths; hence

$$a_{12} = a_{21} = 0. \tag{5.14}$$

For convenience, we also assume that the two groups have the same characteristics,

and that the functions φ_α are constant. Then

$$a_{11} = a_{22} = a > 0, \qquad \varphi_1(x, y) = \varphi_2(x, y) = \frac{k}{L} > 0, \tag{5.15}$$

and, in view of (5.7), the equations (5.8) and (5.13) reduce to

$$\dot{N}_1 = aN_1, \qquad \dot{N}_2 = aN_2, \tag{5.16}$$

and

$$\dot{n}_1 = an_1 - kn_2 + \frac{k}{L} N_2, \qquad \dot{n}_2 = an_2 - kn_1 + \frac{k}{L} N_1. \tag{5.17}$$

Each of (5.16) is the usual population growth equation for a single cultural group; clearly,

$$N_\alpha(t) = \mathring{N}_\alpha e^{at}, \tag{5.18}$$

where $\mathring{N}_\alpha$ is the initial total population of α. If we substitute (5.18) into (5.17), we arrive at a system of ordinary differential equations for n_α ; this system has the solution

$$n_\alpha(t) = e^{at}\left\{ \frac{1}{2}\left[\mathring{n}_\alpha + \mathring{n}_\beta - \frac{1}{L}(\mathring{N}_\alpha + \mathring{N}_\beta) \right]e^{-kt} + \frac{1}{2}\left[\mathring{n}_\alpha - \mathring{n}_\beta - \frac{1}{L}(\mathring{N}_\alpha - \mathring{N}_\beta) \right]e^{kt} + \frac{\mathring{N}_\alpha}{L} \right\}, \tag{5.19}$$

where $\beta \neq \alpha$. Note that we have suppressed the argument x in (5.19); actually, $\mathring{n}_\alpha = \mathring{n}_\alpha(x)$ and $n_\alpha(t) = n_\alpha(x, t)$.

It is clear from (5.19) that $n_\alpha(x, t) \to 0$ in a finite time at a particular point x if and only if the coefficient of e^{kt} is negative:

$$\mathring{n}_\beta(x) - \frac{\mathring{N}_\beta}{L} > \mathring{n}_\alpha(x) - \frac{\mathring{N}_\alpha}{L}. \tag{5.20}$$

Thus at points for which the initial density difference happens to be the same as the average initial density difference, i.e.

$$\mathring{n}_1(x) - \mathring{n}_2(x) = (\mathring{N}_1 - \mathring{N}_2)/L, \tag{5.21}$$

the density of each group will remain positive for all time. On the other hand, at points for which (5.21) is not satisfied, the population of one of the groups will tend to zero. Thus if the initial conditions are such that (5.21) is not satisfied at every point, *then at least one point of the region will become segregated.*

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