A NOTE ON THE STABILITY OF AN IMMISCIBLE LIQUID LAYER IN A POROUS MEDIUM*

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Abstract. The Saffman–Taylor instability of a liquid layer is studied by the standard first-order perturbation method. The viscosity and density of the layer and the bounding fluids are assumed to be constant. Interfacial tension is taken into account. An expression for the rate of growth of perturbations at the interface is determined. Results obtained by previous analyses are shown to be particular cases of this solution. The reduction in the rate of growth of the perturbations due to the presence of the liquid layer is demonstrated.

Introduction. In 1958 Saffman and Taylor [4] considered the stability of an interface between two immiscible superimposed liquids flowing in a porous medium at a constant velocity. Their principal conclusion was that, irrespective of the relative densities of the two liquids, when two superimposed fluids of different viscosities are forced through a porous medium, the interface is unstable if the direction of motion is from the less viscous to the more viscous fluid and stable if the direction of the motion is in the opposite direction. Since that time, a large number of research workers have extended the analysis of the stability of fluid interfaces in porous media (see, for example, [1], [2]). More recently, the analysis of Saffman and Taylor was extended by Raghavan and Marsden [3] to consider the stability of an immiscible liquid layer in porous media. Raghavan and Marsden were principally interested in the mechanism of emulsification in porous media. In their analysis the motion of the liquids on either side of the layer was neglected. This implies that the surrounding fluids have negligible density and mobility (ratio of permeability to viscosity).

The present note is specifically concerned with the instability of a viscous liquid surrounded by two semi-infinite liquids of constant density and viscosity. The interfacial tension between the liquids is taken into account. Results obtained by previous investigators are shown to be special cases of the present analysis.

In the analysis which follows, an expression for the rate of growth of perturbations at the interface is developed. If the interfacial tension is neglected, and the density of the liquids are assumed to be equal, then it is shown that when the viscosity of the intermediate layer lies between the viscosities of the bounding fluids, the principal rate of growth is smaller than that which would occur in the absence of the layer.

The analysis presented not only extends the study of the emulsification process in porous media but also leads to the consideration of another problem of current interest to the oil industry, namely emulsion floods, for it represents to a first approximation

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a fluid varying continuously in viscosity and density. However, the validity of the limiting process would have to be established.

Formulation. The formulation of the problem is essentially identical to the previous analysis made by Raghavan and Marsden. However, in the present instance the mobility and the density of the surrounding liquids are taken into account. The liquids are assumed to be incompressible, to have a constant viscosity, and also to fill the pore spaces of the rock completely under the basic flow conditions. The interfacial tension[1] between the liquids is assumed to be constant. For further details regarding the formulation kindly refer to Raghavan and Marsden.

We will now examine the stability of a liquid layer of density ρ_2 and viscosity μ_2 moving at a constant velocity at the instant it is contained between the planes $z = 0$ and $z = h$. It is bounded by fluids of constant properties. We denote the properties of the fluid in the region $z < 0$ by subscript 1 and that in the region $z > h$ by subscript 3. The interfacial tension at $z = 0$ and $z = h$ are equal and are denoted by σ. We will restrict consideration to the $(x - z)$ plane and assume that the flow is unbounded in the $(x - y)$ plane.

Darcy's law for flow of fluids through porous media may be expressed as

$$\mathbf{V} = -\frac{k\rho}{\mu} \operatorname{grad} \left(zg + \frac{1}{\rho} p \right). \tag{1}$$

Here p is the pressure, z is the vertical distance, $\mathbf{V}$ is the macroscopic velocity, k is the permeability of the medium (which varies for each liquid), and g is the acceleration due to gravity. The equations of motion and continuity may thus be written, respectively, as

$$U = -(k/\mu)(\partial p/\partial x), \tag{2}$$

$$V = -(k/\mu)(\partial(p + \rho g z)/\partial z), \tag{3}$$

$$(\partial U/\partial x) + (\partial V/\partial z) = 0. \tag{4}$$

In the above equations U and V are the velocities in the x and z directions respectively and the equations apply to all three fluids. Let the boundary surfaces be given by

$$z = \eta_j(x, t) \quad (j = 1, 2). \tag{5}$$

The dynamic and kinematic boundary conditions can be written, respectively, as

$$p_k = p_2 - (-1)^j \sigma 2H_j \quad \text{on} \quad z = \eta_j \qquad (j = 1, 2; k = 1, 3), \tag{6}$$

$$\eta_{j,t} - V + U\eta_{j,x} = 0 \quad \text{on} \quad z = \eta_j \qquad (j = 1, 2)$$

where p_k is the pressure in the surrounding liquids, H_j is the curvature of the surface, and the subscripts x, z, t represent partial derivatives with respect to space and time. The initial conditions will be imposed as needed.

Solution. The proposed model assumes that a zero-order flow exists. The first-order perturbation of the same flow satisfies linear equations and is represented by a series

[1] For a discussion of interfacial tension as it relates to this problem, see [1], [2].

or integrals of normal modes. Some of the modes are found to be unstable in the sense that their amplitude increases indefinitely with time.

Zero-order solution. Consider the system to be moving at a constant velocity V^0 in the z direction. Let the bounding surfaces be planes normal to the z axis and independent of x. The solution would then be independent of x and the zero-order velocity is then given by

$$V^0 = \frac{k_2}{\mu_2}\left[\frac{p_2 - p_1}{h} - \rho g\right]. \tag{8}$$

First-order perturbation. Suppose that the initial bounding surfaces differ from planes and that the initial velocity differs slightly from the constant z component assumed in the zero-order unperturbed solution. If now we set u and v as the perturbed velocities, then we can define a stream function as follows:

$$u_k = -(\partial\psi_k/\partial z), \quad v_k = (\partial\psi_k/\partial x) \quad (k = 1, 2, 3). \tag{9}$$

It can be easily shown from (2), (3), (8) and (9) that

$$\nabla^2\psi_k = 0 \quad (k = 1, 2, 3). \tag{10}$$

To solve Laplace's equation for ψ_k let us assume that ψ_k is a product of a function of z and t multiplied by a function of x. Thus let

$$\psi_1 = B \exp(\alpha t) \exp(mz) \cos mx, \tag{11}$$

$$\psi_2 = (C_1 \exp(mz) + C_2 \exp(-mz)) \exp(\alpha t) \cos mx, \tag{12}$$

$$\psi_3 = A \exp(\alpha t) \exp(-mz) \cos mx, \tag{13}$$

where m is the wave number of the disturbance in the z direction, α is the growth factor governing the amplification of the interface, and A, B, C_1, C_2 are constants. It should be noted that ψ_1 and ψ_3 take into account the boundary conditions at $\pm\infty$. The perturbed interfaces may now be prescribed as

$$\eta_1^{\,1}(x, z, t) \equiv \beta \exp(\alpha t) \sin mx - z = 0 \quad \text{at} \quad z = 0, \tag{14}$$

$$\eta_2^{\,1}(x, z, t) \equiv \omega \exp(\alpha t) \sin mx - z + h = 0 \quad \text{at} \quad z = h, \tag{15}$$

where ω and β are arbitrary constants.

The kinematic boundary conditions for the first-order problem may now be written as

$$(\partial\eta_1^{\,1}/\partial t) = \psi_{1,x} = \psi_{2,x} \quad \text{on} \quad z = 0, \tag{16}$$

$$(\partial\eta_2^{\,1}/\partial t) = \psi_{3,x} = \psi_{2,x} \quad \text{on} \quad z = h. \tag{17}$$

The dynamic conditions, using (16) and (17), are as follows:

$$\lambda_2\psi_{2,z}(0) - \lambda_1\psi_{1,z}(0) + \frac{m^2}{\alpha}\{[\rho_2\psi_2(0) - \rho_1\psi_1(0)]g + V[\lambda_2\psi_2(0) - \lambda_1\psi_1(0)]\} = \frac{m^4\sigma}{\alpha}\psi_1(0), \tag{18}$$

$$\lambda_3\psi_{2,z}(h) - \lambda_2\psi_{3,z}(h) + \frac{m^2}{\alpha}\{[\rho_3\psi_3(h) - \rho_2\psi_2(h)]g + V[\lambda_3\psi_3(h) - \lambda_2\psi_2(h)]\} = \frac{m^4\sigma}{\alpha}\psi_3(h), \tag{19}$$

where λ represents the ratio of the viscosity to permeability, *not* the wavelength, and the superscript zero has been dropped on V.

Of prime interest in the analysis is the behavior of α. Eqs. (18) and (19) simplify as follows:

$$[\cosh mh(\lambda_1\lambda_2 + \lambda_2\lambda_3) + \sinh mh(\lambda_2^2 + \lambda_1\lambda_3)]\alpha^2 + m[(B - A)\lambda_2 \cosh mh + (B\lambda_1 - A\lambda_3)\sinh mh]\alpha - m^2AB \sinh mh = 0, \tag{20}$$

where

$$A = (\rho_2 - \rho_1)g + (\lambda_2 - \lambda_1)V - m^2\sigma, \tag{21}$$

$$B = (\rho_2 - \rho_3)g + (\lambda_2 - \lambda_3)V + m^2\sigma. \tag{22}$$

It can be shown that there will be no oscillating motion if

$$[B(\lambda_2 \cosh mh + \lambda_1 \sinh mh) + A(\lambda_3 \sinh mh + \lambda_2 \cosh mh)]^2 \geq 4\lambda_2^2\,|AB|, \tag{23}$$

for $AB \geq 0$.

Now if attention is focused only on modes associated with real exponential time factors, the limiting condition of stability—marginal stability—is given by

$$AB = 0. \tag{24}$$

The above condition can be satisfied for a variety of values of density, viscosity, velocity and interfacial tension. However, for given physical properties, it is evident that the magnitude of the velocity and mobilities play the dominant role as regards to stability. It can also be seen that the interfacial tension has a stabilizing effect on the interface.

The above analysis does not indicate the effect of the presence of the middle layer. As a wide range of possibilities can be considered, we shall examine the specific case in which the densities of the liquids are equal and the interfacial tension between them is neglected. (Note that Saffman and Taylor's conclusions are independent of the relative densities of the liquids.) This case has been chosen so as to simplify the analysis. At the same time the principal features of the problem are included. However, before analyzing this case the limiting forms of (20) will be examined.

Limiting forms of Eq. (20). If surface tension is neglected and if $h \to 0$, we then have the result obtained by Saffman and Taylor,

$$(\lambda_1 + \lambda_3)\alpha = m\{(\lambda_3 - \lambda_1)V + (\rho_3 - \rho_1)g\}. \tag{25}$$

On the other hand, if h is assumed to be finite and the surrounding liquids have a negligible density and mobility we then obtain the following result:

$$\alpha^2 + \frac{2m^3\sigma k_2}{\mu_2} \coth mh \;\; \alpha - \frac{m^2 k_2^2}{\mu_2^2}(\rho_2 g + \lambda_2 V + m^2\sigma)(\rho_2 g + \lambda_2 V - m^2\sigma) = 0. \tag{26}$$

The above equation is identical to that obtained by Raghavan and Marsden [3].

Solution to Eq. (20) assuming equal densities and no interfacial tension. The characteristic equation for this case can be rearranged as

$$\cosh mh\left[\lambda_2(\lambda_1 + \lambda_3) + \lambda_2(\lambda_1 - \lambda_3)\frac{mV}{\alpha}\right]$$

$$+ \sinh mh\left[\lambda_2^2 + \lambda_1\lambda_3 - \lambda_2(\lambda_3 - \lambda_1)\frac{mV}{\alpha} + (\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)\left(\frac{mV}{\alpha}\right)^2\right] = 0. \tag{27}$$

If the wavelength is large compared with the thickness h, the two solutions to the first order in mh are (note that the effects of interfacial tension would not be important)

$$\frac{mV}{\alpha_1} = \frac{\lambda_3 + \lambda_1}{\lambda_3 - \lambda_1} + \frac{4mh\lambda_1\lambda_3(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}{\lambda_2(\lambda_3 - \lambda_1)^3}, \tag{28}$$

$$\frac{mV}{\alpha_2} = \frac{\lambda_2(\lambda_3 - \lambda_1)}{mh(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}. \tag{29}$$

Several points need emphasis. First, even though the above solutions represent an approximation to the actual problem, they do provide insight on the effect of the presence of the middle layer. Second, the above solution includes the case of large mobility differences across the layer—a consideration of great interest in oil-field operations. Third, given values of λ_1 and λ_3, α_1 is at its minimal value when $\lambda_2 = (\lambda_1\lambda_3)^{1/2}$, and fourth, the first term on the right-hand side of (28) can be obtained from (25) by equating the densities in (25).

Fig. 1 gives values of α_1/mV for various values of λ_3/λ_1, λ_2/λ_1 and mh, and Fig. 2 is the percentage reduction in α_1/mV at optimum λ_2. From the figures one can conclude the following:

(i) The principal rate of growth of perturbations at the two interfaces is reduced due to the presence of a layer of mobility λ_2 (which is intermediate between λ_1 and λ_3). The greatest reduction occurs for $\lambda_2 = (\lambda_1\lambda_3)^{1/2}$.

(ii) The reduction in amplification is greatest for small values of λ_3/λ_1 (see Fig. 2).

(iii) For a given mobility ratio λ_3/λ_1 and wavenumber m the amplification rate will decrease as the thickness increases until α_1/mV approaches $\lambda_2 - \lambda_1/\lambda_2 + \lambda_1$.

If we now assume that mh is large then mV/α in (27) may be expanded in powers of 2 mh as follows:

$$\frac{mV}{\alpha_1} = \frac{\lambda_1 + \lambda_2}{\lambda_2 - \lambda_1} + \frac{2\lambda_1\lambda_2(\lambda_3 - \lambda_2)}{(\lambda_1\lambda_3 - \lambda_2^2)(\lambda_2 - \lambda_1)}\exp(-2mh), \tag{30}$$

$$\frac{mV}{\alpha_2} = \frac{\lambda_2 + \lambda_3}{\lambda_3 - \lambda_2} - \frac{2\lambda_2\lambda_3(\lambda_2 - \lambda_1)}{(\lambda_1\lambda_3 - \lambda_2^2)(\lambda_3 - \lambda_2)}\exp(-2mh). \tag{31}$$

Eqs. (30) and (31) show the manner in which the perturbation at an interface is modified by the presence of another interface at a large distance. It is interesting to note that in this instance the amplification factors do not differ by an order of magnitude.

Conclusion. The stability of a liquid layer surrounded by two other liquids has been examined. It has been shown that the velocity and mobility of the liquids play a dominant role in the stability of the system. The results obtained by earlier investigators may be obtained from the analysis presented if appropriate modifications are made. It has been demonstrated that the presence of a middle layer, when the density of the liquids are equal and interfacial tension is neglected, gives rise to two modes whereby both interfaces might become unstable. One of the modes is closely related to the mode which exists in the absence of the layer when the thickness is small compared to the wavelength and is the principal mode causing instability.

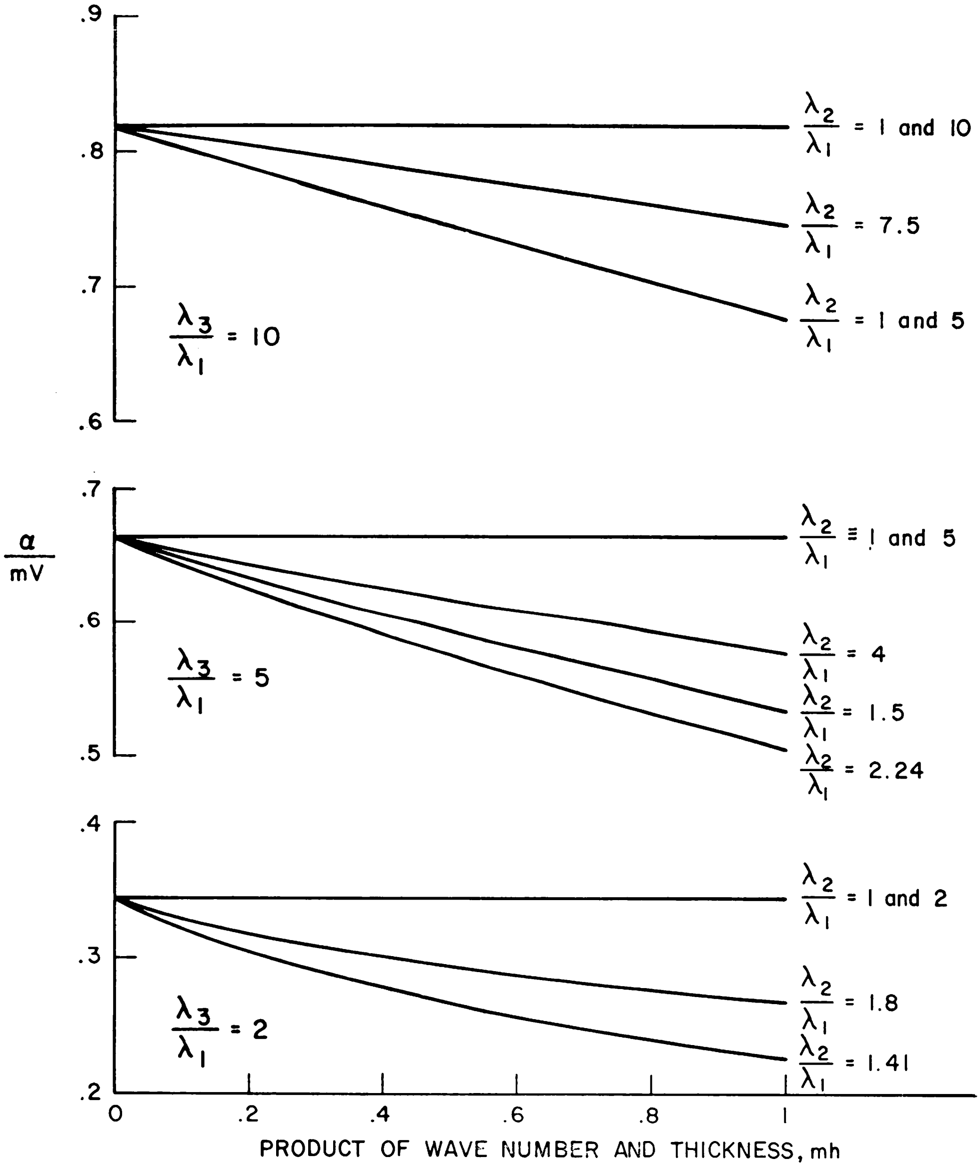

FIG. 1. Calculated principal rate of growth versus wave number-thickness product ($\rho_1 = \rho_2 = \rho_3$ and $\sigma = 0$).

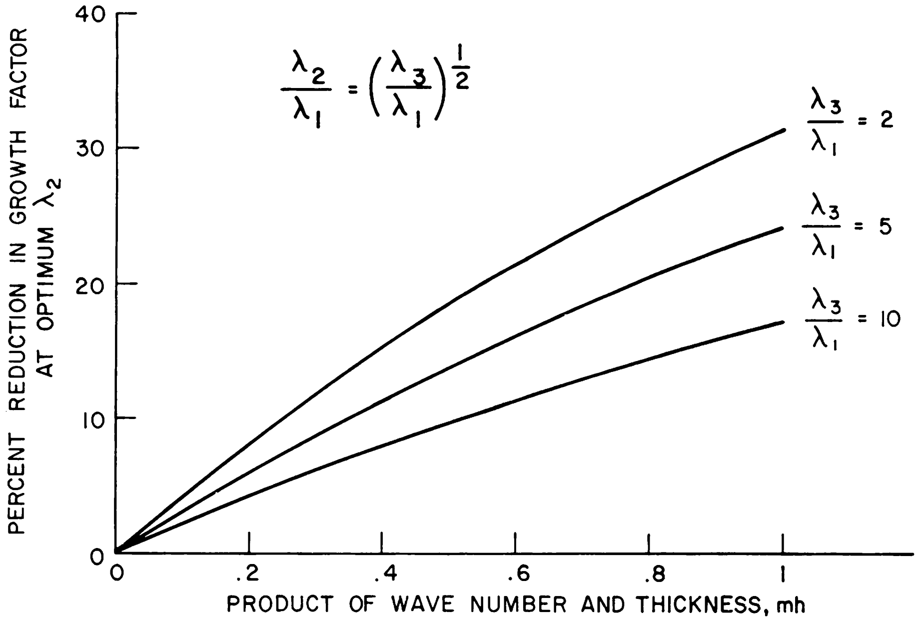

FIG. 2. Calculated reduction in growth factor versus wave number-thickness product ($\rho_1 = \rho_2 = \rho_3$ and $\sigma = 0$).

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