

# QUARTERLY

OF

# APPLIED MATHEMATICS

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# QUARTERLY OF APPLIED MATHEMATICS

The QUARTERLY prints original papers in applied mathematics which have an intimate connection with application in engineering. It is expected that each paper will be of a high scientific standard; that the presentation will be of such character that the paper can be easily read by those to whom it would be of interest; and that the mathematical argument, judged by the standard of the field of application, will be of an advanced character.

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# SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

**Manuscripts:** Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

**Titles:** The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

**Mathematical Work:** As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ('), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent  $\frac{1}{2}$  rather than with the sign  $\sqrt{\phantom{x}}$ .

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp [(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2 + b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos (\pi x / 2 b)}{\cos (\pi a / 2 b)} \text{ is preferable to } \frac{\cos \frac{\pi x}{2 b}}{\cos \frac{\pi a}{2 b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u \, du \text{ is preferable to } \int \frac{\sin u}{u} \, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t(a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

**Cuts:** Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

**Bibliography:** References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (for books)—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; for periodicals—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354–372 (1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten*.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

**Footnotes:** As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

**Abbreviations:** Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq. (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.



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## — BOOK REVIEW SECTION —

*The use of integral transforms.* By Ian H. Sneddon. McGraw-Hill Book Co., New York, 1972. xii + 539 pp. \$21.50.

Professor Sneddon's *Fourier transforms*, published in 1951, was perhaps the first textbook (if we deny that description to Oliver Heaviside's *Electromagnetic theory*) to apply integral transforms to a wide variety of physical problems of (then) current interest. The stated aim of that book was "to present the theory of Fourier transforms and related topics in a form suitable for . . . students and research workers interested in the boundary value problems of physics and engineering." The stated aim of the present book is rather more modest: "to provide an introduction to the use of integral transforms for students of applied mathematics, physics and engineering." The two books are of almost equal length (542 vs. 539 pp.), and both were written for those "whose primary interest is in the applications of the theory rather than in the theory itself"; however, the emphasis and approach of 1951 differ significantly from those of 1972. The differences are exemplified by Sneddon's treatment of Dirac's delta function, which is introduced heuristically in the first chapter and used extensively in 1951 but in 1972 is relegated to the last chapter, where it receives the *de rigueur* foundation of generalized-function theory but is never used; the applied mathematician must judge for himself whether or not this constitutes progress.

An overview of the present book is provided by its table of contents:

1. Introduction (26 pp.); 2. Fourier transforms (108 pp.); 3. The Laplace transform (127 pp., including 10 pp. on Stieltjes and Hilbert transforms); 4. The Mellin transform (36 pp.); 5. Hankel transforms (55 pp.); 6. The Kontorovich-Lebedev transform (16 pp.); 7. The Mehler-Fock transform (54 pp.); 8. Finite transforms (61 pp.); 9. Generalized functions (24 pp.); Appendix A. Bessel functions (8 pp.); Appendix B. Tables of integral transforms (8 pp.).

The coverage of the many types of integral transforms is, to my knowledge, more complete than that in any comparable book. The emphasis on different transforms is, on the whole, well balanced, although (not surprisingly in view of the author's own interests and contributions) there appears to be a relative bias towards transforms that have proved useful in elasticity. (Every reader is likely to have his own merit ranking of integral transforms; I believe that the Hilbert transform deserves more than eight pages vis-à-vis the 54 pp. devoted to the Mehler-Fock transform and find Sneddon's opinion that "The Hilbert transform occurs occasionally in mathematical physics" something of an understatement.)

There is a wide range of applications, including elasticity, fluid mechanics, heat conduction, potential theory, quantum mechanics, statistics, and summation of series, but few of the problems considered are at the research-paper level (which they were in Sneddon's 1951 book). It is perhaps for the latter reason that Sneddon does not mention either the saddle-point or the stationary-phase approximation, both of which appear so frequently in the application of integral transforms to problems in applied mathematics (of the type treated in the 1951 book, where the stationary-phase approximation *is* described and applied). Even more surprising is the omission of an explicit statement of Heaviside's expansion theorem, which provides the inverse of any meromorphic Laplace transform. There are 306 exercises for the student, but only 26 of these are posed in explicitly physical contexts.

Tastes may vary on the amount of historical material that should be included in a textbook of this type, but I would have thought that an exposition that can find space to cite Debnath's papers "On Laguerre transform," "On Jacobi transform," and "On Hermite transform" could have cited Heaviside's *Electromagnetic theory* and Fourier's *The analytical theory of heat*, both of which deal extensively with real physical problems. (Heaviside does not simply pose the problem of heat conduction in a half-space to illustrate the application of his operational calculus to partial differential equations; rather, he discusses Kelvin's problem of the age of the Earth. Fourier's contributions to applied mathematics need no emphasis here.)

Sneddon's exposition is clear and reflects a reasonable compromise between the general and particular. The mathematical level is appropriate for graduate students in science and engineering with



a working knowledge of complex variables. I did not detect any serious errors but did notice a sufficient number of minor slips to suggest careless copy editing and/or proof reading: e.g., in the Bibliography, Söhngen and Wheelon are misspelled, and Morse's *Vibration and Sound* is listed as "Sound and Vibration." The punctuation is inconsistent; e.g., the adjective *boundary-value* is sometimes, but usually not, hyphenated, and I suspect that the author fought a draw with the copy editor in attempting to defy the McGraw-Hill dictum "Omit punctuation following displayed equations" [*The McGraw-Hill author's book* (1955), p. 21].

All in all, this is a useful book that certainly belongs in any large technical library. I have strong reservations, however, as to its value (at \$21.50) as a text book. In introducing his 1951 book, Sneddon tells us that

Books dealing with the applications of integral transforms to physical problems suffer, in the main, from two defects. They seldom attempt to cover the whole field of Fourier transform theory, restricting themselves to the theory of some special transform, such as the Mellin or Laplace transform, which forms but a corner of the entire field. Secondly, the applications discussed are often of a trivial nature—more suitable for illustrating the basic points of the theory than for helping the research worker to acquire a body of technique sufficient to enable him to handle the problems he encounters in the course of his researches.

That book, although perhaps unsuitable as a text in consequence of the absence of exercises for the student, met the first of Sneddon's objections adequately and the second admirably. The present book goes perhaps too far in meeting the first objection but, considering its length, not far enough in meeting the second.

JOHN W. MILES (*La Jolla, California*)

*Approximate calculation of multiple integrals.* By A. H. Stroud. Prentice-Hall, Engelwood N. J., 1971. xiii + 429 pp. \$16.50.

Although a vast number of papers have been written on the subject of numerical integration is, more than one dimension (cubature), A. H. Stroud is the first to write a textbook on this subject. He is very qualified to write a monograph on cubature, since he has probably published more research papers on this subject than anybody else. Stroud has done a fine job of writing this text, in which his own work and the work of others in the field is interrelated, and in which he covers most of the information currently available on cubature.

The book is handy for the student, the expert, and the user. It is simply written and easy to read. The book makes it easy to get to know the problems in the field of cubature, and to find references using the comprehensive bibliography. The comprehensive listing of formulas and computer programs makes it easy to apply existing methods for approximating integrals.

Stroud's monograph may be divided, roughly, into three parts. The first part contains a discussion of the existence and construction of integration formulas and error bounds, the second part consists of formulas and computer programs, and the third part consists of a lengthy bibliography. The book also has a preface, a table of contents, an author index, a subject index, and an index of symbols.

In Chapter I, the introductory chapter, one finds a classification of the notation, a discussion of the contrast between quadrature in one and more than one variable, a discussion of the use of linear transformations, the desirable properties of a formula, and the choice of a formula. One also finds discussions of regions for which formulas are not known, of unsolved problems and of the convergence of a sequence of formulas.

In Chapter II, one finds a general method, as well as a number of special examples, of getting formulas over a region  $R_3$ , where  $R_3$  is the Cartesian product of lower-dimensional regions  $R_1$  and  $R_2$  and where formulas over  $R_1$  and  $R_2$  are known. For example, the product of two formulas over an interval yields a formula over a rectangle, the product of a formula over a circle and a formula over an interval yields a formula over a cylinder, etc.

In Chapters III and IV one finds descriptions of various methods (differing from Cartesian product methods) of constructing cubature formulas.



In Chapter V one finds various methods of obtaining error bounds for formulas, and also a description of constructing cubature formulas by methods based on minimizing the error bounds.

In Chapter VI one finds a discussion of Monte-Carlo, number-theoretic, and quasi-Monte-Carlo methods of evaluating multidimensional integrals. One also finds a discussion of the connection between the discrepancy of a set of points and the error in a quasi-Monte-Carlo method, and also of stratified sampling methods.

In Chapter VII one finds a description of various regions in  $n$ -space for which formulas are given in Chapter VIII. In Chapter IX one finds a description of various regular polytopes in  $n$ -dimensional space. The vertices of these polytopes are the evaluation points of many integration formulas.

In Chapter X the author lists various computer programs for evaluating multidimensional integrals. Included are programs for evaluating iterated integrals, for evaluating integrals over the square, circle, and sphere, the  $n$ -sphere, the circumference of the circle, the surface of the sphere, tori, the triangle, the tetrahedron and  $n$  simplices,  $2 \leq n \leq 8$ , a program for evaluating integrals over the  $n$ -cube by use of Richardson's extrapolation procedure, and a program for the computation of error constants.

The reviewer was very pleased to read this book!

FRANK STENGER (*Montreal*)

*Linear operators, Part III: spectral operators.* By M. Dunford and J. T. Schwartz.  
John Wiley & Sons, New York, 1971. + pp. \$32.50.

This is Part III of the almost definitive work by Dunford and Schwartz on linear operators. It had been the original intention of the authors to include the theory of spectral operators in Part II, but this was not done simply because of their desire to present along with the general theory a number of important (mathematical) applications. The scheme of the book is to present the abstract theory in the text with physical interpretations and discussions of past and present related problems in Notes and Remarks at the end of each chapter. Actually, very little is said about physical applications and interpretations. In order to make it possible to read a major portion of Part III without reference to Parts I and II the concept of a spectral operator, its basic properties, and illustrations are given in Chapter XV (the first chapter of this volume). In the remaining chapters sufficient conditions are given that bounded operators  $T$  be spectral operators, their algebras are studied, a theory for unbounded spectral operators is developed, their perturbations are considered, and lastly spectral operators with continuous spectra are examined. Cauchy's initial value problem inspires in the authors a section in Chapter XV on determinism, which goes quickly from Homer to television and ends on an optimistic note. As this reviewer understands these words, the authors are wrong in concluding that Laplace demonstrated the stability of the solar system. Here Newton may still be right. Certainly this volume, together with Parts I and II, is a prodigious work over which this reviewer marvels and wonders.

J. P. LASALLE (*Providence*)