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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unneccessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ($'$), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\surd$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp\,[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos\,(\pi x/2b)}{\cos\,(\pi a/2b)} \text{ is preferable to } \frac{\cos \dfrac{\pi x}{2b}}{\cos \dfrac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u \, du \text{ is preferable to } \int \frac{\sin u}{u}\, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t(a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (*for books*—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354–372(1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten*.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq. (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.

CONTENTS

—BOOK REVIEW SECTION—

Mathematical optimization and economic theory. By M. D. Intriligator. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1971. xix + 508 pp. $13.95.

This is a textbook in which sections on econmoic questions and institutions alternate with sections on mathematical methods of optimization. The economics is at the level of an intermediate price theory course for undergraduates with extras added from undergraduate courses in mathematical economics. In principle, the student needs no previous knowledge of either mathematics or economics, but some previous exposure to calculus and linear algebra would be clearly helpful. (Appendices on calculus and matrix algebra are a nice review of needed basic mathematics but, of course, no substitutes for previous study.) Following a brief introduction to "economizing" in the first round the techniques of Lagrangian multipliers, the Kuhn-Tucker theorem, the duality principle of linear programming and elementary two-person sum game theory are presented (Part II) and then applied to static economic analysis of household behavior, theory of the firm, industry equilibrium and the welfare economics of general competitive equilibrium (Part III). In a second round, methods of dynamic optimization are developed. Following the statement of the control problem, the student is led through the calculus of variations (Euler equation and transversality conditions), dynamic programming, the maximum principle and differential games. The applications are limited to optimal economic growth for the neoclassical model including the two-sector and heterogeneous capital cases. This reviewer would have preferred a wider range of economic applications, particularly of dynamic programming, to decision problems at the level of the firm.

This is an attractive book. It is of the concise "no-nonsense" variety and leads quickly to the heart of the matter. It may well fill the present need for an undergraduate text in mathematical economics.

MARTIN J. BECKMANN (*Providence, R. I.*)

Statistical tolerance regions: classical and Bayesian. By I. Guttman. Hafner Publishing Co., Darien, Conn., 1970. ix + 150 pp. $7.95.

Here is an up-to-date, comprehensive treatment of the statistical theory of tolerance regions. It is a welcome addition to the literature where there is no other single source which contains even a substantial portion of the relevant material. The author has a very readable style and has done an exceptionally fine job of guiding the reader through the monograph by pointing out the significance of important results as well as their many relationships. Furthermore, the reader is not overburdened with algebra, as the author has made judicious selections regarding the amount of detail.

As for level, this monograph could easily be read by anyone that has completed a standard one-year course in mathematical statistics. For instance, a major method involves the transformation of random variables and then calculation of the new density by finding the Jacobian. The emphasis is on theory and the reader interested in applications is given several references to consult.

Before commenting further, it is well to consider the layout of the monograph. As suggested by the title, it is divided into two parts; the first treats tolerance regions from a sampling theory point of view while the second consists of Bayesian results. Part I has chapters (1) Some needed terminology, notation and definitions (2) Distribution-free tolerance regions (3) Tolerance regions of β-expectation (4) Tolerance regions of β-content (5) Best populations and tolerance regions. Part II consists of (6) Introduction (7) Some needed distribution theory (8) Tolerance regions of β-expectation (9) Tolerance regions of β-content (10) Selection of best populations.

In Chapters 2 and 3, the normal distribution, both univariate and multivariate, and the negative exponential distribution are discussed in detail. There is a wealth of tables, many of them new, pertaining to various combinations of known and unknown parameters in the normal distribution in the univariate case and also for the multivariate normal of dimension ≤ 4. The extensive tables should be useful to the practitioner.

Part II, which constitutes about one-sixth of the monograph, seems to be in keeping with the author's present philosophy of inference and it begins with an unqualified endorsement of Bayesian methods. The work in this part, mainly on the normal distribution, is relatively recent. In contrast, only 3 of the 55 references in Part I appeared after 1960.

The reviewer found the material both interesting and well written and only has the few following minor comments and misprints.

The reader could have benefited from the inclusion of a few more applications which he could keep in mind as the various formulations of the problem are encountered.

Chapters 5 and 10 on best populations are short and the reader would gain a better perspective if he were referred to *Sequential Identification and Ranking Procedures* (1968) by Bechhofer, Kiefer and Sobel.

The monograph starts (p. 3) with a definition of an abstract probability space and seems to say that any measure over a product space is automatically a product measure rather than stating that only those measures will be considered. On the same page, Definition 1.1 contains the undefined term "statistic". It is not the usual definition since it takes values in a sigma field of sets. Also, some measurability condition must be imposed in order to consider certain probabilities which appear in the definition of β-content and β-expectation tolerance regions. Further, Definition 1.2 on page 4 contains the expression $\Pr_\theta$ which is undefined. It is equal to $P_{\mathbf{X}}^{\theta}$ defined on the previous page.

Another slight bit of confusion is caused by the fact that coverages are introduced under the assumption that the population density is continuous but a statement is made after the proof that the procedure can be followed if the population cdf is continuous (see page 195-7B and page 148B). A similar case of terminology appears on page 7, lines 6 and 9.

Misprints noted by the reviewer: Page 5, eqn. (1.3) should read $\Pr_\theta$; Page 8, Theorem 2.1 should read f_η; Page 29, Theorem 2.4, expected value should be zero for every r; Page 144, 6B, g_i, not h_i; Page 150, should read Winkler, R. L. (1967).

In conclusion, the reviewer would strongly recommend this monograph to anyone wishing to learn the current state of knowledge regarding the statistical theory of tolerance regions or to any practioner who needs a handy set of tables for their application.

RICHARD A. JOHNSON (*Madison, Wisconsin*)

Lectures in applications-oriented mathematics. By Bernard Friedman. Edited by Victor Twersky. Holden-Day, Inc., San Francisco, 1969. xi + 257 pp. \$13.95.

This book consists of the revised and edited student notes for eight of the short lecture courses given by the late Professor Bernard Friedman at Sylvania's Electronic Defense Laboratories during the years 1958-1965. The central purpose of these courses was to distill the essentials of various mathematical topics for presentation to an applications-minded audience.

The presentation is mathematical, rather than putting emphasis on applications, but proofs are not stressed. The eight topics covered can be read independently and are not very demanding of the mathematical background on the part of the reader.

Specifically, the topics, in order, are:

1. Distributions: delta functions and distributions, applications of distributions and distributions and transforms.
2. Spectral theory of operators: linear vector spaces and linear operators.
3. Asymptotic methods: asymptotic series, Watson's lemma, methods of steepest descent and of stationary phase, the Airy integral.
4. Difference equations: recurrence relations and difference equations, the method of the generating function, summation by parts and special methods.
5. Complex integration: analytic functions, integrals of rational functions, integrals involving branch points and contour integral representations of solutions to the wave equation.

6. Symbolic methods: differential operators, difference operators, Laplace transforms, symbolic methods and generating functions, non-commutative operators.
7. Probability: elementary properties of probability, infinite sample spaces.
8. Perturbation theory: regular perturbation theory, the Fredholm expansion, examples of singular perturbation problems, singular perturbation procedure.

The number of topics presented in the 257 pages of this book is surprising. Yet a considerable amount is said in each of these topics, in that manner appealing to applications-oriented people which made Professor Friedman's presentations famous. Each of the topics is presented in such a form that it should be well understood by a student who has had a standard advanced calculus course. Needless to say, each of these eight independent presentations is meant as an introduction to the topic. The scientist and engineer truly interested in the use of these mathematical tools will have to go farther than the materials presented in this book permit.

This set of notes is a welcome addition to the literature on mathematical methods for applications, and should be recommended to those practicing engineers and scientists, as well as students, interested in mathematical models and methods.

E. Infante (*Providence, R. I.*)

A book of splines. By Arthur Sard and Sol Weintraub. John Wiley & Sons, Inc., New York, 1971. x + 817 pp. $22.50.

A book of splines, by Arthur Sard and Sol Weintraub, consists primarily of two extensive tables of spline functions and related material. Table 1 is a table of polynomial cardinal splines on a uniform mesh of degrees 3, 5, 7, 9 and 15; Table 2 contains optimal error appraisals pertinent to the use of Table 1. In addition, there are two self-contained expositions of spline functions—Chapters 1 and 2. Chapters 1 is directed to readers desiring to make use of Table 1, but not desiring to delve into the deeper aspects of splines. It does not demand an advanced mathematical background. Chapter 2, on the other hand, covers the theoretical background underlying both Table 1 and Table 2 and is directed at the reader with a more mature mathematical background who desires a deeper understanding of the subject. These expositions provide the knowledge necessary to use Table 1 and Table 2 neither of which can be utilized as easily as, for instance, a table of sines and cosines.

Although *A book of splines* provides useful material not previously available, it is hard to assess its true worth. With the advent of the modern computer the importance of tables has diminished. In addition, splines tend to be utilized in areas where this is particularly true. It is also probably true that most applications are concerned with cubic splines for which there are available simple, numerically well-conditioned algorithms not requiring a uniform mesh. For most applications these algorithms are more efficient than equivalent procedures utilizing cardinal splines. In the cubic case, even when cardinal splines can be used to advantage, they are of a remarkably simple structure. The discovery of this simple structure is due to E. N. Nilsen whose work is referenced by the authors.

J. H. Ahlberg (*Providence, R. I.*)

Handbook of elliptic integrals for engineers and scientists. 2nd ed., revised. By P. F. Byrd and M. T. Friedman. Springer-Verlag, New York, Heidelberg, Berlin, 1970. xvi + 358 pp. $18.50.

The first edition of this book appeared in 1954 and was reviewed in Volume 12 of the Quarterly. The present second edition is essentially a reproduction of the first edition without changes except for correction of all misprints and errors that have been brought to the attention of the authors. The inclusion of additional material on computational methods for efficient calculation of the standard elliptic integrals and elliptic functions had been considered but was deferred in view of the considerable size of the task. References to numerical approximation and computational algorithms are, however, included in a supplementary bibliography which also contains references to textbooks and numerical tables.

This handbook has proven very useful in the past, and in this corrected form is likely to remain the standard work of reference on this subject for a considerable period.

A. ERDÉLYI (*Edinburgh*)

Differential games. By Avner Friedman. John Wiley & Sons, Inc., New York, 1971. *xi* + 350 pp. $17.50.

This book deals with such fundamental questions about differential games as a general definition of strategy, existence of value aad saddle points, capturability, and behavior of the value as a function of the initial data. Except for a final chapter on n-person differential games, zero-sum games with two opposing controllers are considered throughout.

In optimal control theory, the existence problem is to show that there is an admissible control for which the number V = inf (performance criterion) is attained. However, for differential games it is not clear a priori what the corresponding number V = "value" of the differential game should be. Definitions of value and theorems about its existence have been given by the author, and earlier by Varaiya-Lin and the reviewer. The book gives an elegant treatment of these matters. Two numbers V^+, V^-, called the upper and lower values of the differential game, are defined. If V^+, V^- are equal, then their common value V is the value. Sufficient conditions on the game dynamics and payoff functional are given for the existence of V. This is done first for games of fixed duration. (In that case, very recent results of Elliott-Kalton and the author show that V^+, V^- agree with corresponding numbers defined somewhat differently by the reviewer. It follows that a game of fixed duration has a value V if Isaacs' minimax condition holds.)

For games of variable duration, a condition insuring immediate capturability near the terminal set is needed to insure that a value V exists. Without some such condition, there may be, however, a value in the extended sense of Varaiya-Lin.

A separate question is the computation of the value and optimal strategies. See R. Isaacs, *Differential Games*, Wiley, 1965 for a report of his interesting early work on a heuristic basis. Isaacs' method involves the global solution of a first-order partial differential equation (of Hamilton-Jacobi type). In the present book it is shown, under slightly stronger assumptions than those for the existence of V, that Isaacs' equation has a generalized (Lipschitz) solution. Isaacs admits as strategies functions of time and the current state of the game. To prove the existence of V, more complicated strategies depending on past control choices are allowed. It is shown that if the strategies in Isaacs' sense obtained by solving the Hamilton-Jacobi equation are Lipschitz, then they are optimal. Thus the abstract existence theory ties together various heuristic computational results in a rigorous manner.

Further topics included are differential games with restricted phase coordinates, linear-quadratic games, and capturability results (work of Pshenichni).

WENDELL H. FLEMING (*Providence, R. I.*)