A NOTE ON THE POTENTIAL VORTEX IN A WALL JET*

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Abstract. Recently Gurevich [1] found an exact solution for the problem posed by the plane flow past a vortex of an inviscid, incompressible fluid which is bounded above by a free surface and below by a horizontal wall. This solution, which is found by using a conformal mapping of the flow region onto the interior of a semicircular region, is restricted to flows with at least one stagnation point.

The model of a single vortex with circulation such that the force on the vortex is upward is useful in the description of some of the features of the flow past a submerged lifting body; this flow is adequately described by Gurevich's solution. For the case in which the force on the vortex is downward, Gurevich's solution has been extended to include flows without stagnation points but with bifurcation points on the free surface which correspond to singular points of the mapping. This extension describes the flow of a vortex lowered into the fluid from above.

1. Introduction. In studies of a submerged lifting airfoil near a free surface questions have been raised regarding the assumption that the free surface is approximately horizontal and consequently that the free surface boundary condition may be linearized. Therefore, it is of interest to investigate simple models of free-surface flows for which exact nonlinear solutions are obtainable. Gurevich [1] recently found an exact solution for the problem posed by the steady, two-dimensional flow of an inviscid, incompressible fluid past a vortex of strength Γ where the fluid is bounded below by a horizontal wall and above by a free surface; that is, a potential vortex in a wall jet.

At infinity the flow velocity v_0 is uniform and the depth of the fluid is denoted by l. The vortex is located at a depth h below the free surface which is denoted by the unknown function $y = \eta(x)$; see Fig. 1. The flow may be described in terms of the complex potential $\omega = \Phi + i\psi$ where the velocity potential Φ and the stream function ψ are harmonic

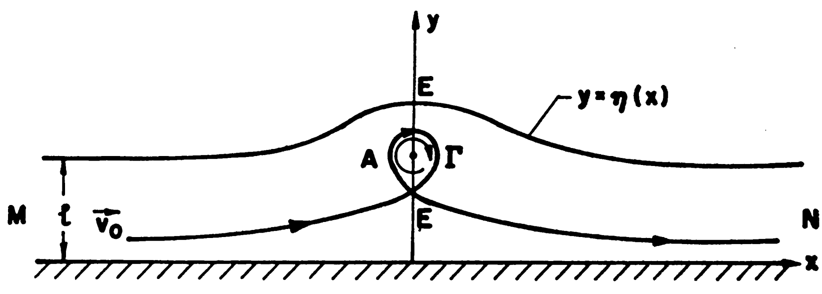

Fig. 1.

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except at the vortex and satisfy the boundary conditions

$$\partial\Phi/\partial y = 0, \qquad \partial\psi/\partial x = 0, \qquad \psi = 0 \quad \text{at} \quad y = 0,$$
$$(\partial\Phi/\partial x)^2 + (\partial\Phi/\partial y)^2 = v_0^2, \qquad \psi = v_0 l \quad \text{at} \quad y = \eta(x).$$

Gurevich mapped the flow region in the $z = x + iy$ plane to the interior of a semi-circular region in the complex τ plane, using Chaplygin's singular point method; see Fig. 2.

The mapping $z(\tau)$ is not known initially but is obtained as part of the solution. The potential $\omega(\tau)$ is constructed in the τ plane from the assumed signularities which produce the flow in the τ plane: a vortex of strength Γ at $\tau = i\gamma$, its images at $\tau = -i\gamma$, $\tau = i/\gamma$ and $\tau = -i/\gamma$, a source of strength $v_0 l$ at $\tau = -1$ and a sink of equal strength at $\tau = 1$. Hence,

$$\omega(\tau) = \frac{2v_0 l}{\pi} \log \frac{1 + \tau}{1 - \tau} + i \frac{\Gamma}{2\pi} \log \frac{\tau - i\gamma}{\tau + i\gamma} \frac{\tau + i/\gamma}{\tau - i/\gamma} \tag{1}$$

with the resulting complex velocity of the flow in the τ plane

$$\frac{d\omega}{d\tau} = \frac{4v_0 l}{\pi}\left[1 - \frac{\Gamma}{4v_0 l}\left(\frac{1}{\gamma} - \gamma\right)\right]\left[\frac{\tau^4 + 2\kappa\tau^2 + 1}{(1 - \tau^2)(\tau^2 + \gamma^2)(\tau^2 + 1/\gamma^2)}\right], \tag{2}$$

where

$$2\kappa = \frac{\gamma^2 + \dfrac{1}{\gamma^2} + \dfrac{\Gamma}{2v_0 l}\left(\dfrac{1}{\gamma} - \gamma\right)}{1 - \dfrac{\Gamma}{4v_0 l}\left(\dfrac{1}{\gamma} - \gamma\right)}. \tag{3}$$

The complex velocity in the z plane, $\zeta(\tau) = (1/v_0)(d\omega/dz)$, is constructed by requiring that $\zeta(\tau)$ has the singularities corresponding to the vortex at $\tau = i\gamma$ and its images—that is, $\zeta(\tau)$ has poles at $\tau = \pm i\gamma$ and zeros at $\tau = \pm i/\gamma$—and that $\zeta(\tau)$ satisfy the boundary, free surface, and infinity conditions:

$$\begin{aligned} \operatorname{Im} \zeta &= 0 \quad \text{on} \quad \operatorname{Im} \tau = 0 \\ |\zeta| &= 1 \quad \text{on} \quad |\tau| = 1 \\ \zeta &= 1 \quad \text{at} \quad \tau = \pm 1. \end{aligned}$$

The mapping $z(\tau)$ is then found by integration of

$$(dz/d\tau) = (1/v_0\zeta(\tau)(d\omega/d\tau). \tag{4}$$

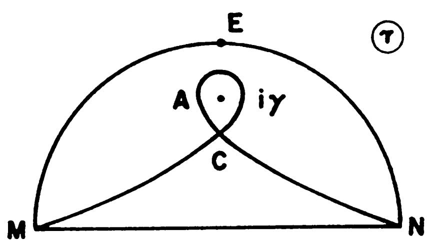

FIG. 2.

2. Summary of Gurevich's solution; $\kappa^2 > 1$. Gurevich required that the mapping be conformal except at $\tau = \pm 1$ and hence that the stagnation points of the flow in the τ plane map into stagnation points of the flow in the z plane. The locations of the stagnation points of the flow in the τ plane are given by the solutions of

$$\tau^4 + 2\kappa\tau^2 + 1 = 0 \tag{5}$$

which has four distinct roots $\tau_{1,2}^2 = -\kappa + (\kappa^2 - 1)^{1/2}$, $\tau_{3,4}^2 = -\kappa - (\kappa^2 - 1)^{1/2}$.

For $\kappa^2 < 1$ the roots of (5) are located on the unit circle; since in steady flow stagnation points can not occur on a free surface, Gurevich restricts his solution to $\kappa^2 > 1$. For $\kappa > 1$ the two critical points inside the unit circle are purely imaginary:

$$\tau_{1,2} = \pm i[\kappa - (\kappa^2 - 1)^{1/2}]^{1/2} = \pm\delta.$$

The critical point corresponding to $\tau_1 = \delta$ is above or below the vortex at $\tau = i\gamma$ depending on the sign of Γ. For $\kappa < -1$ the two roots inside the unit circle are real:

$$\tau_{3,4} = \pm[-\kappa - (\kappa^2 - 1)^{1/2}]^{1/2} = \pm\delta.$$

For the complex velocity in the z plane, Gurevich finds

$$\zeta(\tau) = \frac{(\tau^2 - \delta^2)(\gamma^2\tau^2 + 1)}{(1 - \tau^2\delta^2)(\tau^2 + \gamma^2)}.$$

Integration of (4) yields the mapping

$$z(\tau) = \frac{4v_0 l\, \delta^2\left[1 - \frac{\Gamma}{4v_0 l}\left(\frac{1}{\gamma} - \gamma\right)\right]}{\pi v_0 \gamma^2}\left\{\frac{A}{2}\log\frac{1+\tau}{1-\tau} + B\frac{\gamma^2}{2}\frac{\tau}{\tau^2 + 1/\gamma^2} + \frac{C\gamma}{2i}\log\frac{i/\gamma - \tau}{i/\gamma + \tau}\right\}$$

where (6)

$$A = \frac{(1 - 1/\delta^2)^2}{(1 + 1/\gamma^2)^2}, \qquad B = \frac{(1/\gamma^2 + 1/\delta^2)^2}{(1 + 1/\gamma^2)}, \qquad C = \frac{1/\gamma^2 + 1/\delta^2}{2(1 + 1/\gamma^2)^2}.$$

For $\Gamma/v_0 l > 0$, corresponding to a vortex with clockwise circulation and an upward force, Gurevich's solution adequately describes the flow. With $\Gamma/v_0 l$ fixed for small values of vortex height γ there are two stagnation points on the wall which first separate to a maximum and then coalesce with increasing γ until they coalesce into a single stagnation point. Further increase in γ forces the stagnation point into the interior of the flow; see Fig. 3. The limiting case predicted by Gurevich is *not* a solution, since, in the limiting case, the stagnation point occurs on the free surface. For $\Gamma/v_0 l < 0$, corresponding to a downward force, Gurevich's solution only incompletely describes the flow. Gurevich's solution predicts one stagnation point above the vortex in the interior of the flow and describes the flow when the vortex is near to the wall only up to a critical height determined by the parameter $\Gamma/v_0 l$ and corresponding to raising the stagnation point to the free surface; that is, $\tau = \delta = i$, which limiting case is not a solution. If the vortex is raised above this height, Gurevich's solution does not describe the potential flow that is expected. Alternatively, if the vortex is lowered into the fluid from above, the existing solution does not predict the flow.

3. The solution for $\kappa^2 < 1$. The resolution of this difficulty is to permit the mapping from the z to the τ plane to be non-conformal at the stagnation points in the τ plane when these critical points occur on the semi-circle $|\tau| = 1$. The flow in the z plane will

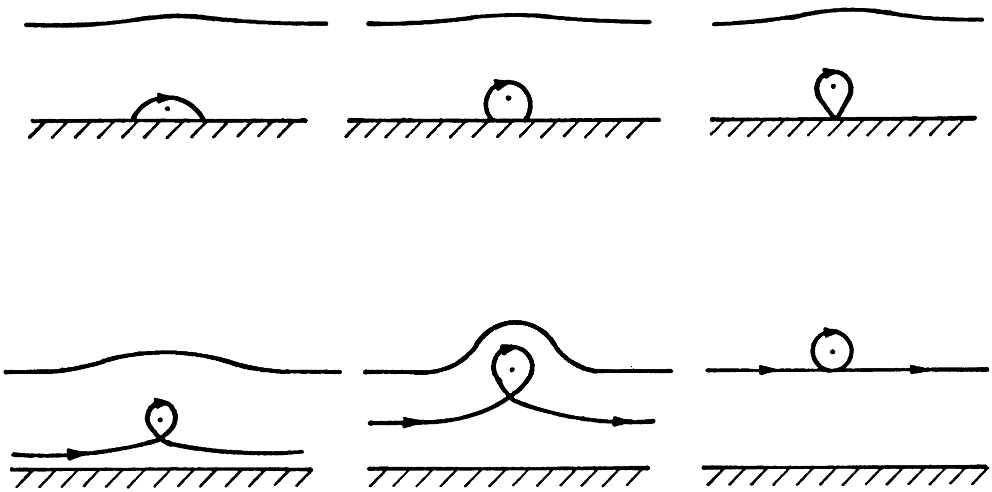

FIG. 3.

have bifurcation points on the free surface where the mapping is non-conformal; see Fig. 4. Potential flows with bifurcation points were first noted by Hopkinson [2].

The complex velocity in the z-plane is constructed without stagnation points, and is found to be

$$\zeta(\tau) = \frac{\gamma^2\tau^2 + 1}{\tau^2 + \gamma^2}$$

and the mapping is the same as (6) with A, B and C replaced by the following quantities:

$$\bar{A} = \frac{2(1+\kappa)}{(1+(1/\gamma^2))^2}, \qquad \bar{B} = \frac{-2(1+\kappa)}{\gamma^2(1+(1/\gamma^2))} + \left(1 + \frac{1}{\gamma^2}\right),$$

$$\bar{C} = \frac{(1+\kappa)(1-(1/\gamma^2))}{(1+(1/\gamma^2))^2} - \tfrac{1}{2}(1-\gamma^2). \tag{7}$$

The flow may be described in terms of a fixed value of $\Gamma/v_0 l$ and varying vortex height; see Fig. 5.

In this case the limiting position corresponding to $\gamma = 1$ and $\kappa = +1$ is a solution with a single bifurcation point; the minimum depth of the vortex below the surface, $h_{\min} = \Gamma/2\pi v_0$, is found to be independent of the depth l of the flow at infinity. As the vortex is lowered the bifurcation point divides into two; these points, denoted by b, occur first on the lower side of the recirculating vortex flow region but, with further

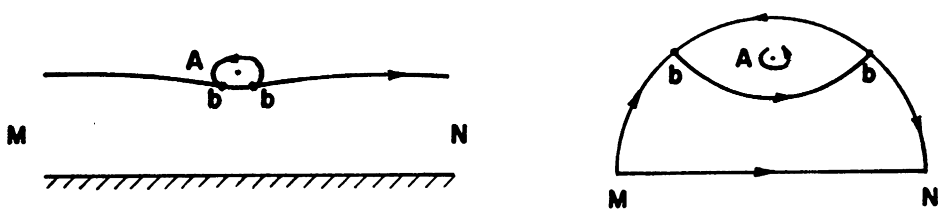

FIG. 4.

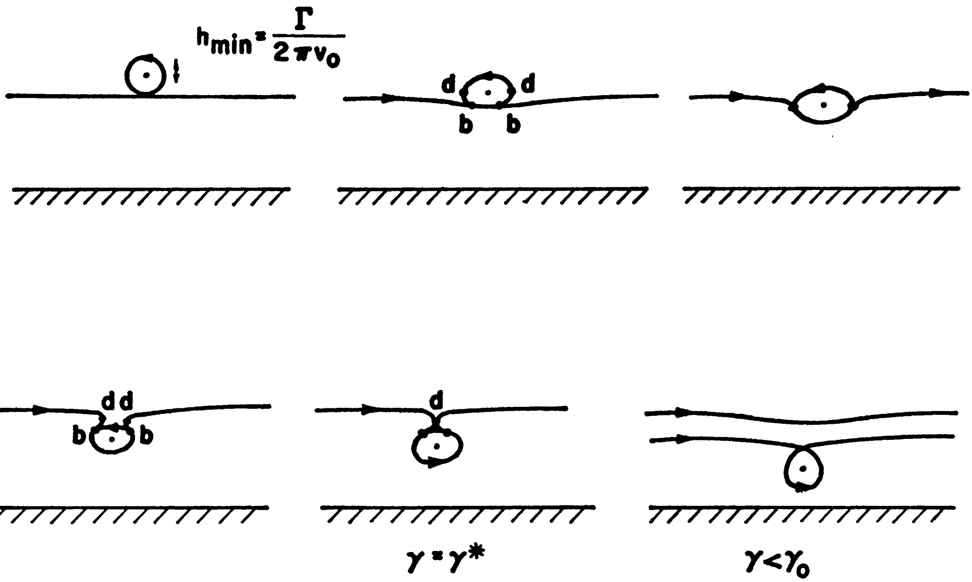

FIG. 5.

lowering of the vortex, then appear on the upper side of the recirculating vortex flow region. The free surface points denoted by d are characterized by vanishing horizontal velocity; these points coalesce above the vortex at a vortex height γ^* found by setting to zero the equation for the x coordinate of the surface points with zero horizontal velocity; there result the transcendental equations

$$\bar{B}^* \frac{\gamma^{*2}}{2} \frac{(\gamma^{*2} + 1) \cos \theta_d}{\gamma^{*2} + \dfrac{1}{\gamma^{*2}} + 2 \cos 2\theta_d} - \frac{\bar{A}^*}{2} \log \tan \frac{\theta d}{2} + \bar{C}^* \frac{\gamma^*}{2} \left[\arctan \frac{\gamma^* \cos \theta_d}{1 - \gamma^* \sin \theta_d} - \arctan \frac{-\gamma^* \cos \theta_d}{1 + \gamma^* \sin \theta_d} \right] = 0$$

where $\cos 2\theta_d = -2/(\gamma^{*2} + (1/\gamma^{*2}))$ and $\bar{A}^*, \bar{B}^*, \bar{C}^*$ are given by (7) evaluated at κ^*, γ^* where κ^* is obtained from (3). Further decrease in γ requires overlapping flows which would persist until γ reached a second limiting critical value

$$\gamma_0 = ((\Gamma/2v_0 l)^2 + 1)^{1/2} + \Gamma/2v_0 l$$

and $\kappa \to 1^-$, corresponding to the limiting case of the coalescence of the bifurcation points. This value, $\gamma = \gamma_0$ and $\kappa \to 1^+$, also corresponds to the limiting case of a stagnation point on the free surface. Hence further decrease in γ submerges the stagnation point and the resulting flow is described by Gurevich's solution.

Then, for each fixed $\Gamma/2v_0 l$, there is an interval of vortex height, $\gamma_0 \leq \gamma \leq \gamma^*$, for which the solution obtained here fails to describe the flow; see Fig. 6.

It is postulated that for this range of vortex height the correct flow description is found by increasing the cavity pressure and thereby decreasing the cavity surface speed until the cavity pressure equals the stagnation pressure and the bifurcation points b

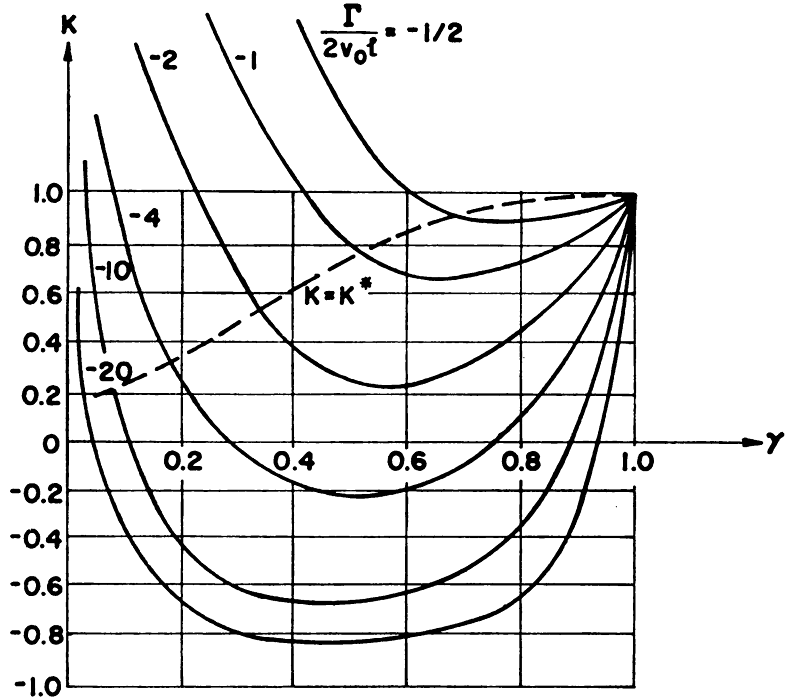

FIG. 6.

coalesce into a stagnation point located at a finite distance below the free surface. This aspect of the flow requires further investigation.

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REFERENCES

[1] M. I. Gurevich, *Vortex near a free surface;* PMM **27** (5), 899–902 (1963); English transl., J. Appl. Math. Mech. **27** (5), 1370–76 (1963)

[2] B. Hopkinson, *On discontinuous fluid motions involving sources and vortices,* Proc. London Math. Soc. **29**, 142–164 (1898)