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FINITE FOURIER SELF-TRANSFORMS*

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Abstract. This paper considers the integral equation

$$\lambda\gamma(\mathbf{t}') = (2\pi)^{-N} \int_\Omega \exp(-i\boldsymbol{\omega}\cdot\mathbf{t}') \int_T \gamma(\mathbf{t}) \exp(i\boldsymbol{\omega}\cdot\mathbf{t})\, d\mathbf{t}\, d\boldsymbol{\omega}$$

as well as a more general one wherein the Fourier kernels are weighted. When Ω and T are N-dimensional spherical domains, the eigenfunctions of the integral equation are generalized prolate spheroidal functions for which a new nomenclature is proposed. Many properties of the eigenfunctions are developed and summarized. Because of the importance of these functions in Fourier transform theory, old as well as new properties are included.

Introduction. In a series of papers Slepian, Landau and Pollak [1, 2, 3, 4] developed many of the properties of a new set of functions which they called generalized prolate spheroidal wave (GPSW) functions. These functions play an important role in the theory of Fourier transforms. Since Fourier transforms are widely used in many fields, the GPSW functions have found many applications, particularly in linear circuit theory and diffraction. [11–18, 23] give a good sampling of such applications; others may be found in [19]. Landau [20] used these functions in a paper on necessary density conditions for sampling and interpolation of certain entire functions. The main mathematical properties of the GPSW that most applications exploit is their double orthogonality, their extremal properties and their usefulness in extrapolating band-limited functions.

The purpose of this paper is to develop and summarize the properties of a class of functions which include the GPSW functions. For reasons explained in Sec. 14 of this paper, we propose a different nomenclature for GPSW functions, namely Slepian functions. The functions considered are solutions of the integral equation

$$\lambda\gamma(\mathbf{t}') = (2\pi)^{-N} \int_\Omega e^{-i\boldsymbol{\omega}\cdot\mathbf{t}'} \int_T \gamma(\mathbf{t}) e^{i\boldsymbol{\omega}\cdot\mathbf{t}}\, d\mathbf{t}\, d\boldsymbol{\omega}.$$

We consider arbitrary and mainly finite domains Ω and T, and the corresponding eigenfunctions are termed finite Fourier self-transforms (FFST for short). Slepian func-

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tions correspond to N-dimensional spherical domains. A more general integral equation is also considered, wherein the Fourier kernels are weighted. Solutions of the general spheroidal equation (not to be confused with the GPSW) are also solutions of such an integral equation. Several practical methods for calculating and using these functions have been developed and will be presented in a subsequent paper.

The contents of the paper are summarized in the following. The notation is contained in Sec. 1. Sec. 2 presents the integral equations whose solutions are the subject of this paper. The double-orthogonality of the FFST and their Fourier transforms, which are also FFST, is considered in Sec. 3. The conditions under which weighted Fourier kernels have solutions with this property are developed and two examples, general spheroidal functions and the bound state wave functions of the square potential well, are cited. Sec. 4 considers domains Ω- and T-symmetric about the origin or some other point, the latter case yielding a theorem analogous to the shifting theorem in Fourier transforms. An important consequence is that the eigenfunctions are of definite parity and are also solutions of the same integral equation but with both Fourier kernels forming a complex symmetric kernel. Sec. 5 shows that if domains Ω and T are the same up to scale factor c ($\Omega = ct$) as well as symmetrical then the integral equation is the iterate of a simpler one. The eigenfunctions corresponding to infinite domains which are the complements of finite ones are treated in Sec. 6. This leads to the consideration in Sec. 7 of the space $\mathcal{E}$ consisting of the union of all T-limited functions and Ω band-limited functions. It is shown that the FFST and their T-limited versions span $\mathcal{E}$ and the projection operator into this space is developed. The missing functions which are needed to span $\mathcal{L}^2$ (the Hilbert space of all square-integrable functions) turns out to be null on T and their Fourier transforms on Ω. The extremal properties of the FFST are reviewed in Sec. 8 and a new one yields a different formulation of the uncertainty principle. Separation of variables is treated in Sec. 9. Cartesian, polar and spherical coordinates are used in Secs. 10 to 12. A synthesis of these coordinate systems is made in Sec. 13 and it is shown that when the domains Ω and T consist of N-dimensional spherical shells, then the problem reduces to the solution of a set of one-dimensional radial integral equations. The differential equation satisfied by GPSW functions is given in Sec. 14 and a change of nomenclature is proposed. It is shown in Sec. 15 that no Sturm–Liouville differential equation can exist when T is a more complicated domain than a simple interval.

1. Notation. Throughout this paper we will adopt the following notation: T, Ω are two finite domains in the N-dimensional euclidian space R^N; $\mathbf{t}$, $\boldsymbol{\omega}$ are N-dimensional real variables: $\boldsymbol{\omega}\cdot\mathbf{t} = \omega_1 t_1 + \omega_2 t_2 \cdots + \omega_N t_N$; $d\mathbf{t} = dt_1\, dt_2 \cdots dt_N$. $\mathfrak{F}f$ and $\mathfrak{F}^{-1}F$ are the Fourier transform and inverse Fourier transform of f and F:

$$\mathfrak{F}f = F(\boldsymbol{\omega}) = (2\pi)^{-N/2} \int_{-\infty}^{\infty} f(\mathbf{t}) e^{i\boldsymbol{\omega}\cdot\mathbf{t}}\, d\mathbf{t},$$

$$\mathfrak{F}^{-1}F = f(\mathbf{t}) = (2\pi)^{-N/2} \int_{-\infty}^{\infty} F(\boldsymbol{\omega}) e^{-i\boldsymbol{\omega}\cdot\mathbf{t}}\, d\boldsymbol{\omega}.$$

$\mathfrak{F}_T f$ is the finite Fourier transform of f:

$$\mathfrak{F}_T f = (2\pi)^{-N/2} \int_T f(\mathbf{t}) e^{i\boldsymbol{\omega}\cdot\mathbf{t}}\, d\mathbf{t}.$$

The inner product and norm over finite and infinite domains are:

$$(f, g) = \int_{-\infty}^{\infty} f(\mathbf{t})g^*(\mathbf{t})\, d\mathbf{t}; \qquad (f, g)_T = \int_T f(\mathbf{t})g^*(\mathbf{t})\, d\mathbf{t};$$

$$\|f\| = (f, f)^{1/2}, \qquad \|f\|_T = (f, f)_T^{1/2}.$$

The characteristic functions $\mathfrak{X}(t)$ are:

$$\mathfrak{X}_T(\mathbf{t}) = 1, \qquad \mathbf{t} \in T$$
$$= 0, \qquad \mathbf{t} \notin T.$$

The time-limiting and band-limiting operators D_T and B_Ω are:

$$\begin{aligned} D_T f(\mathbf{t}) &= f(\mathbf{t}), \qquad \mathbf{t} \in T \\ &= 0, \qquad \mathbf{t} \notin T \\ &= \mathfrak{X}_T f; \\ B_\Omega f &= (2\pi)^{-N} \int_\Omega e^{-i\boldsymbol{\omega}\cdot\mathbf{t}} \int_{-\infty}^{\infty} f(t) e^{i\boldsymbol{\omega}\cdot\mathbf{t}}\, d\mathbf{t}\, d\boldsymbol{\omega} \\ &= \mathfrak{F}^{-1}[\mathfrak{X}_\Omega(\mathfrak{F}f)] = \mathfrak{F}_\Omega^{-1}\mathfrak{F}f. \end{aligned}$$

$\mathfrak{L}^2$ and $\mathfrak{L}_T^2$ are the spaces of all square-integrable functions in $\mathfrak{R}^N$ and T respectively. $\mathfrak{D}_T$ is the space of all T-limited functions, namely all functions f in $\mathfrak{L}^2$ such that $D_T f = f$. $\mathfrak{B}_\Omega$ is the space of all Ω band-limited functions, namely all functions in $\mathfrak{L}^2$ such that $B_\Omega f = f$ or, equivalently, $D_\Omega F = F$.

2. Integral equations. We consider the following integral equation:

$$\lambda\gamma(\mathbf{t}') = (2\pi)^{-N} \int_\Omega e^{-i\boldsymbol{\omega}\cdot\mathbf{t}'} \int_T \gamma(t) e^{i\boldsymbol{\omega}\cdot\mathbf{t}}\, d\mathbf{t}\, d\boldsymbol{\omega}. \tag{2.1}$$

Using abbreviated notation we can write this equation:

$$\lambda\gamma = \mathfrak{F}^{-1}\{\mathfrak{X}_\Omega[\mathfrak{F}(\mathfrak{X}_T\gamma)]\} = \mathfrak{F}_\Omega^{-1}\mathfrak{F}_T\gamma \tag{2.2}$$

or

$$\lambda\gamma = B_\Omega D_T \gamma. \tag{2.3}$$

We will refer to solutions of Eq. (2.1) as finite Fourier self-transforms (FFST for short). The indices of the eigenfunctions and eigenvalues will be written explicitly only when needed. When the indices are shown explicitly, often one index will denote the usual N-tuple index associated with N-dimensional eigenfunctions. By integrating over $\boldsymbol{\omega}$ in Eq. (2.1) we can write:

$$\lambda\gamma(\mathbf{t}') = \int_T \gamma(\mathbf{t}) K(\mathbf{t} - \mathbf{t}')\, d\mathbf{t} \tag{2.4}$$

with the kernel K given by

$$K(\mathbf{t} - \mathbf{t}') = (2\pi)^{-N} \int_\Omega e^{i\boldsymbol{\omega}\cdot(\mathbf{t}-\mathbf{t}')} d\boldsymbol{\omega}. \tag{2.5}$$

This kernel is Hermitian; that is, $K(\mathbf{t} - \mathbf{t}') = K^*(\mathbf{t}' - \mathbf{t})$. It is also positive definite since the expression

$$\int_T f^*(\mathbf{t}') \int_T f(\mathbf{t})K(\mathbf{t} - \mathbf{t}')\, dt\, dt' \tag{2.6}$$

is always positive for any f, as can be seen by using (2.5) for K. The integral equation (2.1) is a special case of the following two integral equations:

$$\lambda f = \mathfrak{F}^{-1}\{\mathcal{Y}[\mathfrak{F}(\mathfrak{X}_T wf)]\}, \tag{2.7}$$

$$\lambda f = w^*\mathfrak{F}^{-1}\{\mathcal{Y}\mathcal{Y}^*[\mathfrak{F}(\mathfrak{X}_T wf)]\}. \tag{2.8}$$

Here $\mathcal{Y}(\omega)$ is a filter function corresponding to the following process: a function $f(t)$ is weighted and time-limited and the result is passed through a filter. Solutions to (2.7) are those functions which reproduce themselves to a constant after being filtered. The resulting function will not of course be time-limited. Those functions whose energy loss after filtering is an extremum are solutions of (2.8). This has been proven by Chalk [11] for the case $w = 1$ and the proof is easily extended to the more general case. If $\mathcal{Y} = \mathcal{Y}\mathcal{Y}^*$ and $w = 1$ then the functions which reproduce themselves after weighting and filtering and those whose energy is an extremum after this operation are the same. This is the case for the FFST since for these functions $w = 1$ and $\mathcal{Y} = \mathfrak{X}_\Omega = \mathcal{Y}\mathcal{Y}^*$.

The integral equations (2.1), (2.7) and (2.8) define $\gamma(t)$ for $t \in T$. Since finite Fourier transforms are analytical, both γ and Γ, defined by

$$\Gamma(\boldsymbol{\omega}) = \mathfrak{F}_T\gamma(\mathbf{t}), \qquad \gamma(\mathbf{t}') = \lambda^{-1}\mathfrak{F}_\Omega^{-1}\Gamma(\boldsymbol{\omega}),$$

are analytical and defined in $\mathcal{R}^N$ inasmuch as $\gamma(t)$ is found for $\mathbf{t} \in T$. This is not necessarily true for Eqs. (2.7) and (2.8). It is the case if $\mathcal{Y}(\boldsymbol{\omega})$ is null outside of some finite domain Ω. More generally the eigenfunctions of (2.7) and (2.8) will not be analytical. In the following it will be assumed that $\mathcal{Y}(\omega)$ is such that the eigenfunctions will be defined for all $\boldsymbol{\omega}$ by (2.7) and (2.8) and will be in $\mathcal{L}^2$.

3. Double orthogonality of the FFST. As has been noted by Slepian [1], perhaps the most striking property of the FFST is their orthogonality over two different domains. A generalization of this is orthogonality with two different weight functions, a property posessed by some solutions of (2.7), (2.8). These properties are described more explicitly below.

Since (2.1) is a Fredholm equation with Hermitian kernel. its solutions are orthogonal over T: $(\gamma_i , \gamma_j)_T = \delta_{ij}$ with the normalization $||\gamma_i||_T = 1$. The γ's are complete in $\mathcal{L}_T^2$. We define the functions Γ to be the finite Fourier transforms of the γ's:

$$\Gamma(\boldsymbol{\omega}) = (2\pi)^{-N/2} \int_T \gamma(\mathbf{t})e^{i\boldsymbol{\omega}\cdot\mathbf{t}}\, dt \tag{3.1}$$

or

$$\Gamma = \mathfrak{F}_T\gamma.$$

By Parseval's theorem the Γ's are orthonormal over $\mathcal{R}^N$: $(\Gamma_i , \Gamma_j) = \delta_{ij}$. They are complete in $\mathcal{B}_T$, the space of T band-limited functions. If we take the finite Fourier transform of (2.1) we obtain:

$$\lambda\Gamma(\boldsymbol{\omega}') = (2\pi)^{-N} \int_T e^{i\boldsymbol{\omega}'\cdot\mathbf{t}} \int_\Omega \Gamma(\boldsymbol{\omega})e^{-i\boldsymbol{\omega}\cdot\mathbf{t}}\, d\omega\, dt. \tag{3.2}$$

This is the same as the original equation with Ω and T exchanged. The Γ's therefore are also FFST, and have the following orthogonality relation:

$$(\Gamma_i , \Gamma_j)_\Omega = \|\Gamma_i\|_\Omega^2 \, \delta_{ij} . \tag{3.3}$$

By inserting the definition (3.1) of Γ into (2.1) we obtain

$$\lambda\gamma = \mathfrak{F}_\Omega^{-1}\Gamma. \tag{3.4}$$

Therefore we also have the following orthogonality relation:

$$(\gamma_i , \gamma_j) = \|\gamma_i\|^2 \, \delta_{ij} . \tag{3.5}$$

It is seen that both sets of FFST are complete in two different spaces: the γ's in $\mathfrak{L}_T^2$ and $\mathfrak{B}_\Omega$, and the Γ's in $\mathfrak{L}_\Omega^2$ and $\mathfrak{B}_T$.

To complete the orthogonality relations (3.3) and (3.5) we shall find the normalization factors on the right of both equations, namely $\|\gamma_i\|^2$ and $\|\Gamma_i\|_\Omega^2$. By taking the inverse Fourier transform of (3.1) we obtain $\mathfrak{X}_T\gamma_i = \mathfrak{F}^{-1}\Gamma_i$; we multiply this by the complex conjugate of (3.4) and integrate to obtain $\lambda_i(\gamma_i , \gamma_i)_T = (\mathfrak{F}^{-1}\Gamma_i , \mathfrak{F}^{-1}\mathfrak{X}_\Omega\Gamma_i)$. By the use of Parseval's theorem we finally obtain: $\|\Gamma_i\|_\Omega^2 = \lambda_i$, since $\|\gamma_i\|_T^2 = 1$. In the same fashion one can show that $\|\gamma_i\|^2 = \lambda_i^{-1}$.

We summarize here the orthogonality properties of the γ's and Γ's as well as the relationships between them:

$$(\gamma_i , \gamma_j) = \lambda_i^{-1} \, \delta_{ij} , \tag{3.6}$$

$$(\gamma_i , \gamma_j)_T = \delta_{ij} , \tag{3.7}$$

$$(\Gamma_i , \Gamma_j) = \delta_{ij} , \tag{3.8}$$

$$(\Gamma_i , \Gamma_j)_\Omega = \lambda_i \, \delta_{ij} , \tag{3.9}$$

$$\lambda_i\gamma_i = \mathfrak{F}^{-1}\mathfrak{X}_\Omega\Gamma_i , \tag{3.10}$$

$$\mathfrak{X}_T\gamma_i = \mathfrak{F}^{-1}\Gamma_i , \tag{3.11}$$

$$\Gamma_i = \mathfrak{F}\mathfrak{X}_T\gamma_i , \tag{3.12}$$

$$\mathfrak{X}_\Omega\Gamma_i = \lambda_i\mathfrak{F}\gamma_i . \tag{3.13}$$

Solutions of (2.7) possess similar orthogonality properties. Let $\bar{\gamma}_i$ be a solution of this equation and let $\bar{\Gamma}_i$ be its weighted finite Fourier transform, namely $\bar{\Gamma}_i = \mathfrak{F}_T w\bar{\gamma}_i$. Then in a similar fashion as for the FFST one can obtain the following relations:

$$(w\bar{\gamma}_i , \bar{\gamma}_j)_T = \delta_{ij} , \tag{3.14}$$

$$(\mathfrak{Y}\bar{\Gamma}_i , \bar{\Gamma}_j) = \lambda_i \, \delta_{ij} . \tag{3.15}$$

The first relation follows from the fact that (2.7) is an integral equation in polar form. Its kernel can be transformed into a Hermitian one by the transformation $\bar{\gamma}' = w^{1/2}\bar{\gamma}$. The second relation is obtained by noting that $(\mathfrak{Y}\bar{\Gamma}_i , \bar{\Gamma}_j) = (\lambda_i\mathfrak{F}\bar{\gamma}_i , \mathfrak{F}\mathfrak{X}_T w\bar{\gamma}_j) = \lambda_i(w\bar{\gamma}_i , \bar{\gamma}_j)_T$. If $w = 1$ we will have double orthogonality for the $\bar{\Gamma}$'s:

$$(\bar{\Gamma}_i , \bar{\Gamma}_j) = (\bar{\gamma}_i , \bar{\gamma}_j)_T = \delta_{ij} , \tag{3.16}$$

and if $\mathfrak{Y} = \mathfrak{X}_\Omega$ the $\bar{\gamma}$'s will be doubly orthogonal:

$$(\bar{\gamma}_i , \bar{\gamma}_j) = \lambda_i^{-1}\lambda_j^{-1}(\bar{\Gamma}_i , \bar{\Gamma}_j)_\Omega = \lambda_i^{-1} \, \delta_{ij} . \tag{3.17}$$

In a similar fashion it is seen that solutions of (2.8) and their weighted finite Fourier transforms possess the orthogonality relations

$$(\bar{\gamma}_i , \bar{\gamma}_j)_T = \delta_{ij} , \tag{3.18}$$

$$(\mathcal{Y}\Gamma_i , \mathcal{Y}\Gamma_j) = \lambda_i \delta_{ij} . \tag{3.19}$$

The same double orthogonality as for solutions of (2.7) will hold if $w = 1$ or $\mathcal{Y} = \mathfrak{X}_\Omega$. It should be noted that solutions of either (2.7) or (2.8) can be doubly orthogonal even though $w \neq 1$ and $\mathcal{Y} \neq \mathfrak{X}_\Omega$. The condition to be met is one of commutation of two operators. Consider solutions of (2.8) and their inner product over the domain A. The latter will be given by:

$$\begin{aligned}(\gamma_i , \gamma_j)_A &= \lambda_i^{-1}\lambda_j^{-1} \int_A \int_T \gamma(t)K(t'', t)\, dt \int_T \gamma^*(t')K^*(t'', t')\, dt'\, dt'' \\ &= \lambda_i^{-1}\lambda_j^{-1} \int_T \gamma(t) \int_T \gamma^*(t')H_A(t', t)\, dt'\, dt \\ &= \lambda_i^{-1}\lambda_j^{-1} \int_T \gamma(t)M_A\gamma^*(t')\, dt,\end{aligned} \tag{3.20}$$

where K is the kernel of (2.8) and H_A is given by $H_A(t', t) = \int_A K(t'', t)K^*(t'', t')\, dt''$. Clearly the γ's are eigenfunctions of the operator M_T. In order that they be orthogonal over A as well, the operators M_A and M_T must commute, which means that we must have:

$$\int_T H_A(t'', t') \int_T H_T(t', t)f(t)\, dt\, dt' = \int_T H_T(t'', t') \int_T H_A(t', t)f(t)\, dt' \tag{3.21}$$

where f is any function in $\mathfrak{L}_T^2$. This commutation is easily verified for the FFST. Another important case where this applies is the integral equation

$$\lambda\phi(t') = \int_{-1}^{1} \phi(t)(1 - t'^2)^{a/2}(1 - t^2)^{a/2} \int_{-c}^{c} (c^2 - \omega^2)^a e^{i\omega(t-t')}\, d\omega\, dt. \tag{3.22}$$

This equation is seen to be of the same form as (2.8) with $w = (1 - t^2)^{a/2}$, $\mathcal{Y} = (c^2 - \omega^2)^{a/2}\mathfrak{X}_\Omega(\omega)$ and $\Omega = (-c, c)$. Its solutions are weighted general spheroidal functions, namely $\phi = (1 - t^2)^{-a/2} S(t)$ where S is a solution of the general spheroidal differential equation

$$(1 - t^2)S'' - 2(a + 1)S' + (b - c^2t^2)S = 0 \tag{3.23}$$

and which for special values of the parameter a reduces to spheroidal wave functions and Mathieu functions. Their double orthogonality has been demonstrated by Rhodes [5, 21] who used a Sturmian proof which is considerably simpler than the commutation of the operators.

Another interesting case was considered by Chalk [11]. The search for optimum pulse shapes for communication channels led him to consider the integral equation (2.8) with $w = 1$, $\mathcal{Y} = (1 + \omega^2/c^2)^{-1/2}$, $T = (-1, 1)$ and $\Omega = (-\infty, \infty)$. The solutions are elementary functions

$$\begin{aligned}\gamma(t) &= \cos a_n t, \qquad \sin a_n t & |t| &< 1, \\ \gamma(t) &= \pm\cos a_n e^{-a_n|t|}, \sin a_n e^{-a_n|t|} & |t| &> 1,\end{aligned} \tag{3.24}$$

where a_n is a solution of:

$$\operatorname{tg} a_n = c/a_n \qquad (\text{even } \gamma\text{'s}),$$

$$\operatorname{ctg} a_n = -c/a_n \qquad (\text{odd } \gamma\text{'s}).$$

Referring to [13–16, 19], we see that the finite Fourier transforms of γ

$$\Gamma = \tfrac{1}{2}(2\pi)^{-1/2}\left(\frac{\sin (a_n - \omega)}{a_n - \omega} \pm \frac{\sin (a_n + \omega)}{a_n + \omega}\right) \tag{3.25}$$

are doubly orthogonal. These functions appear to be the only elementary functions possessing such a property. It may be noted that these functions represent the bound state eigenfunctions of the square well potential in quantum mechanics and the transverse field distribution in one-dimensional dielectric waveguides.

4. Symmetry considerations. If in Eq. (2.4) we replace $\mathbf{t}$ by $-\mathbf{t}$ we obtain:

$$\lambda_i\gamma_i(\mathbf{t}') = \int_T \gamma_i(-\mathbf{t})K(\mathbf{t} + \mathbf{t}')\, d\mathbf{t}. \tag{4.1}$$

If the γ's are either even or odd functions of $\mathbf{t}$, then they also satisfy the integral equation with the complex symmetric kernel $K(\mathbf{t} + \mathbf{t}')$. The only difference will be that the eigenvalues corresponding to odd γ's will become negative. The interest of the kernel $K(\mathbf{t} + \mathbf{t}')$ is that it appears often in practical applications, electromagnetic resonator theory and iterated diffraction in particular.

A necessary and sufficient condition for the γ's to be odd or even is that the domains T and Ω be symmetric about the origin: that is, T is symmetric if $\mathbf{t} \in T$ implies $-\mathbf{t} \in T$. If we change variables in (2.1) by replacing $\mathbf{t}$, $\mathbf{t}'$, and $\boldsymbol{\omega}$ by their negatives, the kernel remains unchanged as well as the limits of integration if Ω and T are symmetric. Therefore $\gamma(\mathbf{t})$ and $\gamma(-\mathbf{t})$ are both solutions corresponding to the same eigenvalue λ. If $\gamma(\mathbf{t})$ is not even or odd then there will be two independent solutions, one even and one odd, corresponding to that eigenvalue, namely $\frac{1}{2}[\gamma(\mathbf{t}) + \gamma(-\mathbf{t})]$ and $\frac{1}{2}[\gamma(\mathbf{t}) - \gamma(-\mathbf{t})]$.

To see that the symmetry condition is sufficient consider any Ω band-limited function $f(t)$:

$$f(\mathbf{t}) = \int_\Omega F(\boldsymbol{\omega})e^{i\boldsymbol{\omega}\cdot\mathbf{t}}\, d\boldsymbol{\omega}. \tag{4.2}$$

If f is even then perform the change of variable $\mathbf{t}' = -\mathbf{t}$ and subtract from 4.2 to obtain:

$$\int_{-\infty}^{\infty} \mathfrak{X}_\Omega(\boldsymbol{\omega})F(\boldsymbol{\omega}) \sin \boldsymbol{\omega}\cdot\mathbf{t}\, d\boldsymbol{\omega} = 0. \tag{4.3}$$

This expression is zero for all $\mathbf{t}$ if and only if $\mathfrak{X}_\Omega(\boldsymbol{\omega})F(\boldsymbol{\omega})$ is an even function. This implies that F is even and Ω symmetric. The latter condition is of course unnecessary if $F(\boldsymbol{\omega}) = \mathfrak{X}_{\Omega'}(\boldsymbol{\omega})G(\boldsymbol{\omega})$, where Ω' is symmetric and $\Omega' \subset \Omega$. Similar reasoning shows that F must be even and Ω symmetric if f is to be odd.

Now since $\lambda\gamma$ is the Ω-limited inverse Fourier transform of the function $\mathfrak{F}_T\gamma$, Ω must be symmetric and $\mathfrak{F}_T\gamma$ of definite parity if γ is to be of definite parity. The domain T must also be symmetric since $\mathfrak{F}_T\gamma$ is required to be of definite parity.

To summarize, all the eigenfunctions are of definite parity if and only if both Ω and T are symmetric about the origin. If one or both domains Ω and T is asymmetric, then no eigenfunctions of definite parity exist.

The FFST corresponding to domains Ω and T which are not symmetric about the origin but about some other point are related in a simple fashion to those which are. Let Ω' and T' be two domains obtained by shifting the domains Ω and T by amounts $\mathbf{t}_0$ and $\boldsymbol{\omega}_0$; that is, if $\mathbf{t} \in T'$ then $\mathbf{t} - \mathbf{t}_0 \in T$. If $\gamma(\mathbf{t})$ is a FFST corresponding to T and Ω, then

$$\gamma'(\mathbf{t}) = \gamma(\mathbf{t} - \mathbf{t}_0) \exp(-i\boldsymbol{\omega}_0 \cdot \mathbf{t}) \tag{4.4}$$

is a FFST corresponding to T' and Ω'. This can be verified directly by inserting this function into the integral equation (2.1) and making the change of variables $\mathbf{t}'' = \mathbf{t} - \mathbf{t}_0$, $\mathbf{t}''' = \mathbf{t}' - \mathbf{t}_0$ and $\boldsymbol{\omega}' = \boldsymbol{\omega} - \boldsymbol{\omega}_0$. Therefore if either of the domains T or Ω can be simplified by a shift in origin, this should be done before proceeding to a solution.

5. The case $\Omega = cT$. When the domains T and Ω are the same up to a scale factor c, we can write $\Omega = cT$, meaning that $\mathbf{t} \in T$ if and only if $c\mathbf{t} \in \Omega$. If we make the change of variable $\boldsymbol{\omega}' = \boldsymbol{\omega}/c$ then (2.1) becomes

$$\lambda c^{-N}\gamma(\mathbf{t}') = (2\pi)^{-N} \int_T \gamma(\mathbf{t}) \int_T e^{ic\boldsymbol{\omega}'\cdot(\mathbf{t}-\mathbf{t}')} \, d\boldsymbol{\omega}' \, d\mathbf{t}. \tag{5.1}$$

If T is a symmetric domain then the γ's will also be a solution of

$$\pm\lambda c^{-N}\gamma(\mathbf{t}') = (2\pi)^{-N} \int_T \gamma(\mathbf{t}) \int_T e^{ic\boldsymbol{\omega}\cdot(\mathbf{t}+\mathbf{t}')} \, d\boldsymbol{\omega} \, d\mathbf{t}, \tag{5.2}$$

as was shown in the preceding section. This integral equation is the first iterate of:

$$\mu\gamma(\boldsymbol{\omega}) = (2\pi)^{-N/2} \int_T \gamma(\mathbf{t}) e^{ic\boldsymbol{\omega}\cdot\mathbf{t}} \, d\mathbf{t}. \tag{5.3}$$

The eigenvalues μ are related to the λ's through $\lambda = \pm\mu^2 c^N$. The Γ's are now the same functions as the γ's up to a scale factor $\Gamma(\boldsymbol{\omega}) = \mu\gamma(\boldsymbol{\omega}/c)$. It should be emphasized that the reduction to the simpler integral equation (5.3) is feasible only if T is symmetric.

Similar remarks hold for Eqs. (2.7) and (2.8). In particular, it is easily seen that solutions of (3.22) are also solutions of:

$$\mu\gamma(\omega/c) = (2\pi)^{-1/2}(c^2 - \omega^2)^{a/2} \int_{-1}^{1} \gamma(t)(1 - t^1)^{a/2} e^{i\omega t} \, dt. \tag{5.4}$$

6. Domains Ω and T of infinite measure. Solutions of (2.1) corresponding to domains of infinite measure which are the complements of finite domains may be expressed in terms of the eigenfunctions of the latter. Specifically, we will consider the domains Ω' and T' given by $T' = \mathcal{R}^N - T$ and $\Omega' = \mathcal{R}^N - \Omega$ where Ω and T are finite. Solutions corresponding to the latter have been considered in the previous sections. For clarity's sake, the dependence of the eigenfunctions and eigenvalues on the domains Ω and T will be made explicit in places.

We shall consider the following three integral equations:

$$B_{\Omega'} D_T \gamma(\Omega', T) = \lambda(\Omega', T)\gamma(\Omega', T), \tag{6.1}$$

$$B_{\Omega} D_{T'} \gamma(\Omega, T') = \lambda(\Omega, T')\gamma(\Omega, T'), \tag{6.2}$$

$$B_{\Omega'} D_{T'} \gamma(\Omega', T') = \lambda(\Omega', T')\gamma(\Omega', T'). \tag{6.3}$$

The solutions can be found in terms of the functions $\gamma(\Omega, T)$ and eigenvalues $\lambda(\Omega, T)$.

We shall also find the T- or T'-limited Fourier transforms Γ of the γ's. The normalization will be the same as the one used previously, namely $\|\gamma\|_{T \text{ or } T'} = 1$. The choice of T or T' depends on whether D_T or $D_{T'}$ is used in the integral equation.

The operators $B_{\Omega'}$ and $D_{T'}$ are given by $B_{\Omega'} = I - B_\Omega$ and $D_{T'} = I - D_T$, as can be verified through their definition. Solutions to all three equations can be found by assuming them to be of the form $aD_T\gamma(\Omega, T) + b\gamma(\Omega, T)$ and solving for a and b. With the convention that in the right-hand side of the following equation $\gamma = \gamma(\Omega, T)$ and $\lambda = \lambda(\Omega, T)$, the results are:

$$\gamma(\Omega', T) = (1 - \lambda)^{-1}(D_T\gamma - \lambda\gamma), \tag{6.4}$$

$$\lambda(\Omega', T) = 1 - \lambda, \tag{6.5}$$

$$\Gamma(\Omega', T) = \Gamma, \tag{6.6}$$

$$\gamma(\Omega, T') = \lambda^{1/2}(1 - \lambda)^{-1/2}\gamma, \tag{6.7}$$

$$\lambda(\Omega, T') = 1 - \lambda, \tag{6.8}$$

$$\Gamma(\Omega, T') = \lambda^{1/2}(1 - \lambda)^{-1/2}(\lambda^{-1}D_\Omega\Gamma - \Gamma), \tag{6.9}$$

$$\gamma(\Omega', T') = [\lambda(1 - \lambda)]^{-1/2}(D_\Gamma\gamma - \lambda\gamma), \tag{6.10}$$

$$\lambda(\Omega', T') = \lambda, \tag{6.11}$$

$$\Gamma(\Omega', T') = [\lambda(1 - \lambda)]^{-1/2}(\lambda\Gamma - D_\Omega\Gamma). \tag{6.12}$$

The question arises whether other solutions exist to (6.1), (6.2), (6.3). Clearly functions f such that $D_T f = f$ are trivial solutions of (6.2), (6.3) with $\lambda = 0$, while functions f such that $D_{T'} f = f$ are trivial solutions of (6.1). From the fact that the $\gamma(\Omega, T)$'s are complete in $\mathfrak{L}_T^2$ and in $\mathfrak{B}_\Omega$ one can deduce that all the solutions of (6.1), (6.2) have been found. However, any functions in $\mathfrak{L}^2$ which are zero on T and have a Fourier transform zero on Ω are solutions of (6.3) with $\lambda(\Omega', T') = 1$, since these conditions imply $D_{T'} f = f$ and $B_{\Omega'} f = f$. The set of all these functions forms a closed space which is described in the next section.

7. The space $\mathfrak{D}_T + \mathfrak{B}_\Omega$ and its complement. In the preceding sections it was found that the functions $D\gamma$ and γ span the spaces $\mathfrak{D}_T$ and $\mathfrak{B}_\Omega$. Both spaces are closed and orthogonal to their complements, $\mathfrak{L}^2 - \mathfrak{D}_T$ and $\mathfrak{L}^2 - \mathfrak{B}_\Omega$. Furthermore, they form a least angle between them [2], namely $\theta_m = \cos^{-1} \lambda_0^{1/2}$, where the angle is defined by $\theta(d, b) = \cos^{-1} (\operatorname{Re} (d, b)/\|d\| \, \|b\|)$ and $d \in \mathfrak{D}_T$, $b \in \mathfrak{B}_\Omega$. Let us now consider the space $\mathcal{E} = \mathfrak{D}_T \cup \mathfrak{B}_\Omega$ and its complement $\mathcal{E}' = \mathfrak{L}^2 - \mathcal{E}$. Since $\mathcal{E}'$ can also be written in the form $\mathcal{E}' = \mathfrak{L}^2 - (\mathfrak{B}_\Omega \cup \mathfrak{D}_T)$ we see that $\mathcal{E}' = \mathfrak{B}_{\Omega'} \cap \mathfrak{D}_{T'}$, where the primes denote the complements of the spaces. This formulation shows that $\mathcal{E}'$ contains all such f for which $D'f = f$ and $B'f = f$, namely all such functions which are null on T and whose Fourier transforms are null on Ω. There do not seem to be any known functions which possess this property. However, such functions must clearly exist since $\mathcal{E} \neq \mathfrak{L}^2$ and hence $\mathcal{E}'$ is not empty.

The easiest way to construct such functions is to use the projection operator H' into $\mathcal{E}'$ given by $H' = I - H$, where H is the projection operator into $\mathcal{E}$. H can be expressed in terms of the operators B and D:

$$H = \sum_{m=0}^{\infty} (A_m + C_m) \tag{7.1}$$

where $A_m = (1 - B)(DB)^m D$, and $C_m = (1 - D)(BD)^m B$. This operator has previously been used by Landau and Pollak [2]. It can be verified that H is a projection operator by developing H^2:

$$H^2 = \sum_{m,n=0}^{\infty} (A_m A_n + A_m C_n + C_m C_n + C_m A_n).$$

Direct expansion shows that

$$A_m A_n = A_{m+n} - A_{m+n+1}, \qquad C_m C_n = C_{m+n} - C_{m+n+1}, \qquad A_m C_n = C_m A_n = 0;$$

therefore:

$$H^2 = \sum_{m,n=0}^{\infty} (A_{m+n} - A_{m+n+1} + C_{m+n} - C_{m+n+1}).$$

By putting $m' = m + n$ we obtain:

$$H^2 = \sum_{m=0}^{\infty} \sum_{m'=m}^{\infty} (A_{m'} - A_{m'+1} + C_{m'} - C_{m'+1}).$$

For $m = k$, only two terms in the sum over m' do not cancel, namely $A_k + C_k$ and the end result is $H^2 = \sum_m (A_m + C_m) = H$. To show that H projects into $\mathcal{E}$ it suffices to show that if $f \in \mathcal{E}$ then $Hf = f$ and if $f \in \mathcal{E}'$ then $Hf = 0$. Any f in $\mathcal{E}$ is of the form $f = Bg + Dh$ where $g, h \in \mathcal{L}^2$, and use of (7.1) verifies that $Hf = f$. If $f \in \mathcal{E}'$ then $f = B'f = D'f$, and again use of (7.1) shows that $Hf = 0$ since $BB' = DD' = 0$.

A practical method for using H is the following. If $f \in \mathcal{L}^2$ then form the series $Df = \sum a_i D\gamma_i$, $Bf = \sum b_i \gamma_i$. Using (7.1) we will have:

$$Hf = \sum_{m,i} [(1 - B)(DB)^m a_i D\gamma_i + (1 - D)(BD)^m b_i \gamma_i]. \tag{7.2}$$

Since $BD\gamma = \lambda\gamma$ it follows that $(BD)^m \gamma = \lambda^m \gamma$ and (7.2) becomes

$$Hf = \sum_{m,i} [a_i \lambda_i^m [D\gamma_i - \lambda\gamma_i] + b_i \lambda_i^m [\gamma_i - D\gamma_i]]. \tag{7.3}$$

Now $\sum_m \lambda_i^m = (1 - \lambda_i)^{-1}$, since $0 < \lambda < 1$, and we finally obtain

$$Hf = \sum_i [(a_i - b_i)(1 - \lambda_i)^{-1} D\gamma_i + (b_i - \lambda_i a_i)(1 - \lambda_i)^{-1} \gamma_i].$$

A complete set of functions f_i' forming a basis for $\mathcal{E}'$ is given by $f_i' = (I - H)f_i$, where f_i is a basis for $\mathcal{L}^2$. The three sets of functions $\{D\gamma_i\}$, $\{\gamma_i - D\gamma_i\}$ and $\{f_i\}$ together span $\mathcal{L}^2$. The situation is analogous to one considered by Bergman [22] who constructed functions of a complex variable orthogonal over two domains A_1 and A_2 in the complex plane such that $A_1 \subset A_2$. These functions were complete in A_1 but not in A_2. The functions missing to form a complete set in A_2 were null on A_1. In our case the set $D\gamma$ is complete in $\mathcal{L}_T^2$. By adding the sets $\{\gamma - D\gamma\}$ and $\{f'\}$, functions null on T, we obtain a complete set for $\mathcal{L}^2$.

8. Extremal properties of the FFST. Much of the interest of the FFST stems from their extremal properties. Several papers have utilized these properties to solve problems in apodization [12, 13], antenna radiation pattern synthesis [14], and optimum pulse calculation in data transmission [11].

It is well known that the solutions of $Mf = \lambda f$, where M is a Hermitian operator, are those functions which are extrema of the quantity $E = (f, Mf)\,||f||^{-2}$. The maximum and minimum of E correspond to the solutions with largest and smallest eigenvalue, and furthermore $\lambda = E$. All other solutions give rise to E's which are either local minima and maxima or analogues of inflection points. From these considerations we see that since the γ's are solutions of $BD\gamma = \lambda\gamma$, they are extrema of $(f, BDf)\,||f||^{-2}$. In particular it is seen that $D_T\gamma_0$ is the function in $\mathcal{L}^2$ which loses the least energy after being first T-limited and than band-limited. The loss of energy is $1 - \lambda_0$ where λ_0 is the largest eigenvalue of (2.1). An equivalent formulation of this fact is to say that $D\gamma_0$ is the function of norm one in $\mathcal{L}_T^2$ which maximizes the energy in Ω of its Fourier transform, namely $||\mathfrak{F}D\gamma_0||_\Omega^2$, and that this energy is equal to λ_0. Since $DB(D\gamma) = \lambda D\gamma$ it is also seen that $D\gamma_0$ is the function in $\mathcal{L}^2$ which loses the least energy after first being band-limited and then T-limited. Again, an equivalent statement is to say that γ_0 is the band-limited function which concentrates the most energy in T. All of these properties have already been noted by Slepian and Pollak.

Perhaps the most interesting extremal property of the FFST is the one which permits a formulation of the uncertainty principle. Let f be a function in $\mathcal{L}^2$ of norm one and having the Fourier transform F. The uncertainty principle used in quantum mechanics states that:

$$(f, [t - t_0]^2 f) \times (F, [\omega - \omega_0]^2 F) \geq \tfrac{1}{2}. \tag{8.1}$$

While this result is very important in quantum mechanics, it is less useful in the general theory of Fourier transforms since the inner products in (8.1) do not always converge. This is the case for $f(t) = \sin t/t$. A more useful method of measuring the concentration of energy of f and F is to use the following quantities:

$$\alpha^2 = ||f||_T^2 = (f, D_T f), \tag{8.2}$$

$$\beta^2 = ||F||_\Omega^2 = (F, D_\Omega F). \tag{8.3}$$

By Parseval's theorem (8.3) is also given by:

$$\beta^2 = (f, B_\Omega f). \tag{8.4}$$

Landau and Pollak [2] have considered the quantities α^2, β^2 and their application to the uncertainty principle. They found the following relation:

$$\cos^{-1}\alpha + \cos^{-1}\beta \geq \cos^{-1}\lambda_0^{1/2}, \tag{8.5}$$

where again λ_0 is the largest eigenvalue of $B_\Omega D_T\gamma = \lambda\gamma$. This inequality permits one to find the greatest β^2 given α^2 when Ω and T are specified. Conversely, if α^2 and β^2 are given, then (8.5) tests whether these values are compatible with given domains Ω and T. By taking the cosine of (8.5) one obtains the algebraic inequality

$$\alpha\beta - [(1 - \alpha^2)(1 - \beta^2)]^{1/2} \leq \lambda_0^{1/2}, \tag{8.6}$$

or, by first transferring $\cos^{-1}\beta$ to the right of (8.5),

$$\alpha \geq \beta\lambda_0^{1/2} + [(1 - \beta^2)(1 - \lambda_0)]^{1/2}. \tag{8.7}$$

Now consider the quantities $E_1 = (\alpha^2 + \beta^2)/2$ and $E_2 = \alpha^2\beta^2$. Both are measures of the simultaneous concentration of f in T and F in Ω. It is thus of interest to find their maximum values. We first present the result, and then the proofs.

$$(\alpha^2 + \beta^2)/2 \leq (1 + \lambda_0^{1/2})/2, \tag{8.8}$$

$$\alpha^2\beta^2 \leq (1 + \lambda_0^{1/2})^2/4. \tag{8.9}$$

Equality holds for the function f_{opt} :

$$f_{\text{opt}} = [2(1 + \lambda_0^{1/2})]^{1/2}(D_T\gamma_0 + \lambda_0^{1/2}\gamma_0) \tag{8.10}$$

with the normalization $||f_{\text{opt}}|| = 1$. Its Fourier transform F_{opt} is given by:

$$F_{\text{opt}} = [2(1 + \lambda_0^{1/2})]^{1/2}[\lambda_0^{-1/2}D_\Omega\Gamma_0 + \Gamma_0]. \tag{8.11}$$

Furthermore, for the optimum function f_{opt} we have:

$$\alpha^2 = \beta^2 = (1 + \lambda_0^{1/2})/2. \tag{8.12}$$

The easiest path towards a proof is to see that for those values α and β which maximize either E_1 or E_2, the equality sign in (8.5) must hold. To prove this, suppose that α and β are values for which E_1 or E_2 is maximum, but that $\cos^{-1}\alpha + \cos^{-1}\beta > \cos^{-1}\lambda_0^{1/2}$. Then it is possible to find a $\beta' > \beta$ so that (8.5) holds, since $\cos^{-1}\beta$ is a monotonically decreasing function of β. Clearly we will then have $(\alpha^2 + \beta'^2) > (\alpha^2 + \beta^2)$ and $\alpha^2\beta'^2 > \alpha^2\beta^2$, which contradicts the hypothesis that α and β maximize E_1 or E_2. A similar argument will show that the same values of α and β maximize both E_1 and E_2.

Since Eq. (8.5) and the definitions of E_1 and E_2 are symmetric in α and β, this suggests that the maximum occurs for $\alpha^2 = \beta^2$. This is in fact the case. Denote by α_m, β_m those values of α, β giving rise to the maximum value E_{1m} of E_1. We have that $\alpha_m^2 + \beta_m^2 = 2E_{1m}$ and $\alpha = g(\beta)$, where g is the right-hand side of inequality (8.7). It is seen that α_m satisfies $\alpha_m^2 + g^2(\alpha_m) = 2E_{1m}$. If more than one distinct root exists, pick the greatest one between zero and one. But β_m satisfies exactly the same equation, namely $\beta_m^2 + g(\beta_m^2) = 2E_{1m}$, and hence $\alpha_m^2 = \beta_m^2$. Using this fact in Eq. (8.5), we have $2\cos^{-1}\alpha_m = \cos^{-1}\lambda_0^{1/2}$, which yields $\alpha_m^2 = (1 + \lambda_0^{1/2})/2$.

It may also be noted that $\alpha^2 + \beta^2 = (f, B_\Omega + D_T f)$. The eigenfunctions of the operator $B_\Omega + D_T$ are given by $(D_T\gamma_i \pm \lambda_i^{1/2}\gamma_i)$ and its eigenvalues by $1 \pm \lambda_i^{1/2}$. This shows that the extrema of E_1 are $(1 \pm \lambda_i^{1/2})/2$.

To find all the extrema of $\alpha^2\beta^2$ is somewhat lengthier. One method is to expand the arbitrary function in the series

$$f = \sum a_i D\gamma_i + \sum b_i\gamma_i + \sum c_i f_i', \tag{8.12}$$

where f_i' is a complete orthonormal set in $\mathcal{E}' = \mathcal{L}^2 - (\mathcal{D}_T + \mathcal{B}_\Omega)$. It is clear that $a_i = b_i = 0$ is an extremum, since $\alpha^2 = \beta^2 = 0$ for this case. This means that any function in $\mathcal{E}'$ is a trivial extermum with $E_1 = E_2 = 0$. Therefore put $c_i = 0$ in (8.12). Now α^2, β^2 and $||f||^2$ will be given by:

$$\alpha^2 = ||D_Tf||^2 = \sum a_i^2 + b_i^2 + 2a_ib_i, \tag{8.13}$$

$$\beta^2 = ||D_\Omega f||^2 = \sum a_i^2\lambda_i + 2a_ib_i + b^2\lambda^{-1}, \tag{8.14}$$

$$||f||^2 = \sum a_i^2 + 2a_ib_i + b_i^2\lambda_i^{-1}. \tag{8.15}$$

In order for f to be an extremum we must have $\partial v/\partial a_k = \partial v/\partial b_k = 0$ for all k, where $v = \alpha^2\beta^2 - C\,||f||^2$ and C is a Lagrangian multiplier. The derivatives give rise to the equations:

$$\alpha^2(\partial\beta^2/\partial a_k) + \beta^2(\partial\alpha^2/\partial a_k) = C(\partial\,||f||^2/\partial a_k),$$
$$\alpha^2(\partial\beta^2/\partial b_k) + \beta^2(\partial\alpha^2/\partial b_k) = C(\partial\,||f||^2/\partial b_k)$$

or

$$\alpha^2(a_k\lambda_k + b_k) + \beta^2(a_k + b_k) = C(a_k + b_k), \tag{8.16}$$

$$\alpha^2(a_k + b_k\lambda_k^{-1}) + \beta^2(a_k + b_k) = C(a_k + b_k\lambda_k^{-1}). \tag{8.17}$$

We have obtained an infinite set of pairs of equations for α^2 and β^2. Let α_i^2 and β_i^2 denote a solution to these equations. The constant C may depend on i but not on k. Application of Cramer's rule to the kth pair of equations yields:

$$\alpha_i^2 = Cb_k(1 - \lambda_k^{-1})/A_k\,, \qquad \beta_i^2 = Ca_k(\lambda_k - 1)(a_k + b_k\lambda_k^{-1})/A_k(a_k + b_k),$$

where $A_k = a_k(\lambda_k - 1) + b_k(1 - \lambda_k^{-1})$. Since C may not vary with k, the only way to satisfy all equations is to set all coefficients a_k, b_k to zero except for one index, $k = i$. By using Eqs. (8.13), (8.14) relating α^2, β^2 to a_k, b_k and simplifying, one finally obtains $a_k = \pm\lambda_k^{-1/2}b_k$. Therefore the functions $D\gamma_i \pm \lambda_i^{1/2}\gamma_i$ are also extrema of $\alpha^2\beta^2$ with $\alpha_i^2\beta_i^2$ given by $(1 \pm \lambda_i^{1/2})/4$. It may also be noted that these functions span $\mathfrak{D}_T + \mathfrak{B}_\Omega$ and are infinite Fourier self-transforms if $\Omega = T$.

9. Separation of variables in Cartesian coordinates. Solutions of (2.1) can be written as a product, $\gamma(\mathbf{t}) = \prod_i^N \gamma^i(t_i)$, if the domains Ω and T are such that the characteristic functions $\mathfrak{X}_\Omega$ and $\mathfrak{X}_T$ can be written as a product: $\mathfrak{X}_T(\mathbf{t}) = \prod_i^N \mathfrak{X}_i(t_i)$. For such domains the limits of integration in the N-tuple integrals can be written:

$$\int_T = \int_{T_1}\int_{T_2}\cdots\int_{T_N}\,,$$

where T_i are one-dimensional intervals of integration containing no variables. The kernel (2.5) of (2.1) can then also be written as a product: $K(\mathbf{t} - \mathbf{t}') = \prod_i K_i(t_i - t_i)$ with

$$K_i(t_i - t_i') = \frac{1}{2\pi}\int_{\Omega_i} e^{i\omega_i(t_i - t_i')}\,d\omega_i\,. \tag{9.1}$$

It can easily be verified that if $\gamma(t)$ is a solution of

$$\lambda^i\gamma^i(t') = \int_{T_i}\gamma^i(t_i)K_i(t_i - t_i')\,dt_i \tag{9.2}$$

then $\prod_i^N \gamma^i$ is a solution of (2.1) with eigenvalue $\prod_i^N \lambda^i$. The same remarks hold for solutions of (2.7) and (2.8) if $\mathcal{Y}(\omega)$ can be written as a product.

10. Polar coordinates. If Ω and T are either circular or annular domains, then the angular part of the solutions of (2.1) can be found directly, leaving one-dimensional radial integral equations. We denote by Ω_ρ and T_r the one-dimensional domains consisting of the intervals which are bounded by the inner and outer radii of the annular domains of Ω and T respectively. By changing to polar coordinates:

$$t_1 = r\cos\theta, \qquad t_2 = r\sin\theta,$$
$$\omega_1 = \rho\sin\phi, \qquad \omega_2 = \rho\sin\phi,$$

the kernel $K(\mathbf{t} - \mathbf{t}')$ of (2.1) will be given by:

$$K(r, r', \theta, \theta') = (2\pi)^{-2} \int_0^{2\pi} \int_{\Omega_\rho} e^{i\rho[r \cos(\theta-\phi) - r' \cos(\theta'-\phi)]} \rho \, d\rho \, d\phi. \tag{10.1}$$

Using the series

$$e^{ix \cos\alpha} = \sum_{-\infty}^{\infty} i^n J_n(x) e^{in\alpha} \tag{10.2}$$

and integrating over ϕ, we obtain:

$$K(r, r', \theta, \theta') = (2\pi)^{-1} \sum_n \left[\int_{\Omega_\rho} J_n(r\rho) J_n(r'\rho) \rho \, d\rho \right] e^{in(\theta-\theta')}. \tag{10.3}$$

The integral over ρ can be found in closed form:

$$D_n(r\rho, r'\rho) = \int^\rho J_n(r\rho') J_n(r'\rho') \rho' \, d\rho', \tag{10.4}$$

with D_n given by

$$D_n(r\rho, r'\rho) = \left[\frac{\rho}{r^2 - r'^2} [r J_{n+1}(r\rho) J_n(r'\rho) - r' J_{n+1}(r'\rho) J_n(r\rho)] \right]. \tag{10.5}$$

The function D_n with $n = 0$ has previously been used by Lansraux and Boivin [12] for the determination of encircled energy relative to a diffraction pattern of revolution.

The following notation will be used:

$$[D_n(\rho r, \rho r')]_{\Omega_\rho} = \sum_i D_n(\rho r, \rho r')\big|_{\rho=a_i}^{b_i},$$

where a_i and b_i are the inner and outer radii of the annular regions of Ω.

The complete integral equation can now be written:

$$\lambda\gamma(r', \theta') = (2\pi)^{-1} \sum_n \int_0^{2\pi} \int_{T_r} \gamma(r, \theta)[D_n(r\rho, r'\rho)]_{\Omega_\rho} e^{in\theta} r \, dr \, d\theta \, e^{-in\theta'}. \tag{10.6}$$

If we now expand γ as a Fourier series:

$$\gamma(r, \theta) = \sum_m a_m e^{-im\theta}$$

we find that $a_m(r) = \hat{\gamma}_{mn}(r)$ and $\lambda = \lambda_{mn}$ where $\hat{\gamma}_{mn}$ and λ_{mn} are solutions of:

$$\lambda_{mn}\hat{\gamma}_{mn}(r') = \int_{T_r} \hat{\gamma}_{mn}(r)[D_m(r\rho, r'\rho)]_{\Omega_\rho} r \, dr \tag{10.7}$$

Through the use of the definition of D_m (10.4) and the identity $J_n = (-1)^n J_{-n}$ we see that $D_m = D_{-m}$. Therefore $\hat{\gamma}_{-m,n} = \hat{\gamma}_{m,n}$ and only positive indices m need be used. The complete eigenfunctions will then be expressed in terms of the radial eigenfunctions:

$$\gamma_{mn}(r, \theta) = \hat{\gamma}_{|m|n}(r) e^{-im\theta}.$$

11. Spherical coordinates. If the domains Ω and T are either spherical or composed of spherical shells, the angular part of the solution of (2.1) can again be found explicitly. The procedure is essentially the same as for polar coordinates. The spherical coordinates used are:

$$t_1 = r \sin\theta \cos\phi, \qquad t_2 = r \sin\theta \sin\phi, \qquad t_3 = r \cos\theta,$$

$$\omega_1 = \rho \sin\alpha \cos\beta, \qquad \omega_2 = \rho \sin\alpha \sin\beta, \qquad \omega_3 = \rho \cos\alpha.$$

The Fourier kernel can be expanded in spherical harmonics [10]:

$$e^{i\omega\cdot t} = 4\pi \sum_{l=0} \sum_{m-l}^{l} i^l \mathcal{Y}_l^{m*}(a, \beta)\mathcal{Y}_l^m(\theta, \phi)\pi^{1/2} \; (2r\rho)^{-1/2} J_{l+1/2}(r\rho), \tag{11.1}$$

where $\mathcal{Y}_l^m$ is a spherical harmonic:

$$\mathcal{Y}_l^m(\theta, \phi) = \left(\frac{(2l+1)(l-m)}{4\pi(l+m)}\right)^{1/2} P_l^m(\cos\theta)e^{im\phi}. \tag{11.2}$$

These harmonics are orthonormal:

$$\int_{-\pi}^{\pi} \int_0^{2\pi} \mathcal{Y}_l^{m*}(\alpha, \beta)\mathcal{Y}_{l'}^{m'}(\alpha, \beta) \sin\alpha \, d\alpha \, d\beta = \delta_{mm'} \, \delta_{ll'} \, . \tag{11.3}$$

Putting the expansion (11.1) into expression (2.5) for the kernel and integrating over the angular variables, we obtain:

$$K(\mathbf{t} - \mathbf{t}') = \sum_{lm} \int_{\Omega_\rho} \mathcal{Y}_l^m(\theta, \phi)\mathcal{Y}_l^{m*}(\theta', \phi')J_{l+1/2}(\rho r) \; J_{l+1/2}(\rho r')(rr')^{-1/2}\rho \, d\rho \tag{11.4}$$

$$= \sum_{ml} \mathcal{Y}_l^m(\theta, \phi)\mathcal{Y}_l^{m*}(\theta', \phi')[D_{l+1/2}(\rho r, \rho r')]_{\Omega_\rho}(rr')^{-1/2}, \tag{11.5}$$

where $D_{l+1/2}$ has previously been defined by (10.4).

If we now expand the solutions of (2.1) in spherical harmonics:

$$\gamma(r, \theta, \phi) = \sum_l \sum_{m=-l}^{l} a_l^m(r)\mathcal{Y}_l^{m*}(\theta, \phi),$$

it is seen that $a_l^m(r) = \hat{\gamma}_{lm}^m(r)$ and $\lambda = \lambda_{lm}^m$ where $\hat{\gamma}$ and λ_{lm}^m are solutions of

$$\lambda_{ln}^m \hat{\gamma}_{ln}^m(r') = \int_{T_r} \hat{\gamma}_{ln}^m(r)[D_{l+1/2}(\rho r, \rho r')]_{\Omega_\rho}(rr')^{-1/2} r^2 \, dr. \tag{11.6}$$

Furthermore, $\hat{\gamma}_{ln}^m = \hat{\gamma}_{ln}^{m'}$, $\lambda_{ln}^m = \lambda_{ln}^{m'}$, and therefore the index m is not needed for the radial functions $\hat{\gamma}$. The complete eigenfunctions of (2.1) can then be expressed in terms of the radial eigenfunctions

$$\gamma_{ln}^m(r, \theta, \phi) = \hat{\gamma}_{ln}(r)\mathcal{Y}_l^{m*}(\theta, \phi).$$

12. Synthesis of the three coordinate systems. Solutions of (2.1) in the three coordinate systems can be expressed in terms of eigenfunctions of the same one-dimensional integral equation if Ω and T are domains symmetric about the origin. The domains we considered in polar and spherical coordinates already possess this symmetry. Let us consider the one-dimensional case of (2.1):

$$\lambda\gamma(t') = (2\pi)^{-1} \int_T \gamma(t) \int_\Omega e^{i\omega(t-t')} \, d\omega \, dt. \tag{12.1}$$

Using the identity

$$e^{i\omega t} = 2^{-1/2}(\pi\omega t)^{1/2}[J_{-1/2}(\omega t) + iJ_{1/2}(\omega t)], \tag{12.2}$$

we obtain, for the kernel of (12.1):

$$K(t - t') = \tfrac{1}{4}(tt')^{1/2} \int_\Omega [J_{-1/2}(\omega t)J_{-1/2}(\omega t') + J_{1/2}(\omega t')J_{1/2}(\omega t) + iJ_{1/2}(\omega t)J_{-1/2}(\omega t') - iJ_{-1/2}(\omega t)J_{1/2}(\omega t')]\omega\, d\omega. \qquad (12.3)$$

If Ω is symmetric about the origin then the terms $J_{-1/2}J_{1/2}$ will drop out in the integration, since multiplied by ω they will become odd. The kernel will then become:

$$K(t - t') = \tfrac{1}{2}(tt)^{1/2}[D_{-1/2}(\omega t, \omega t') + D_{1/2}(\omega t, \omega t')]_{\Omega+} \qquad (12.4)$$

where $\Omega+$ is the positive part of the domain Ω. If T is also symmetric about the origin, then the γ's will be either even or odd and solutions of

$$\lambda_{2n}\gamma_{2n}(t') = \int_{T+} \gamma_{2n}(t)[D_{-1/2}(\omega t, \omega t')]_{\Omega+}(tt')^{1/2}\, dt \qquad (12.5)$$

for the even γ's, and

$$-\lambda_{2n+1}\gamma_{2n+1}(t') = \int_{T+} \gamma_{2n+1}(t)[D_{1/2}(\omega t, \omega t')]_{\Omega+}(tt')^{1/2}\, dt \qquad (12.6)$$

for the odd γ's.

Turning now to the radial equation (10.7) in polar coordinates and writing $\phi_{mn}(r) = r^{1/2}\hat{\gamma}_{mn}(r)$, we obtain an equation for the ϕ_{mn} :

$$\lambda_{mn}\phi_{mn}(r') = \int_{T_r} \phi_{mn}(r)[D_m(r\rho, r\rho')]_{\Omega_\rho}(rr')^{1/2}\, dr \qquad (12.7)$$

which is the same as (12.6) except for the index m. The same thing is true of the radial equation (11.6) in spherical coordinates. By writing $\phi_{mn}(r) = r\hat{\gamma}_{mn}(r)$ we obtain:

$$\lambda_{ln}\phi_{ln}(r) = \int_{T_r} \phi_{ln}(r)[D_{l+1/2}(r\rho, r'\rho)]_{\Omega_\rho}(rr')^{1/2}\, dr. \qquad (12.8)$$

The reason why the three coordinate systems yield the same kernel is that all three are special cases of the N-dimensional spherical coordinate system. This system is described by Erdelyi [6]. In one dimension there is a positive variable r and a discrete angular variable θ, taking only the values $\pm\pi$, corresponding to positive or negative values of the cartesian variable x. For dimensions higher than 3, (2.1) would again reduce to radial equations with kernels involving the functions $D_{(N+2l-2)/2}$ with $l = 0, 1, 2 \cdots$, except in one dimension where $l = 0, 1$. This is shown by Slepian [4].

We shall henceforth concern ourselves mainly with the solutions of

$$\lambda_{\nu n}\phi_{\nu n}(r') = \int_{T_r} \phi_{\nu n}(r)[D_\nu(r\rho, r'\rho)]_{\Omega_\rho}(rr')^{1/2}\, dr \qquad (12.9)$$

for $\nu \geq -\frac{1}{2}$.

13. The case $\Omega = cT$ for the radial integral equation. It has been shown in Sec. 4 that when $\Omega = cT$ and both domains are symmetric about the origin, then solutions of (2.1) are also solutions of a simpler integral equation. If we make the change of variable $\rho' = \rho/c$ in (12.9) we obtain:

$$\lambda_{\nu n}\phi_{\nu n}(r') = \int_{T_r} \phi_{\nu n}(r)[D_\nu(c\rho r, c\rho r')]_{T_r}(rr')^{1/2}\, dr. \qquad (13.1)$$

This equation is seen to be the first iterate of:

$$\mu_{\nu n}\phi_{\nu n}(r') = \int_{T_r} \phi_{\nu n}(r)J_\nu(crr')(crr')^{1/2}\, dr, \tag{13.2}$$

and $c\mu_{\nu n}^2 = \lambda_{\nu n}$. Eq. (13.2) can also be written as a finite Hankel transform of order ν:

$$\mu_{\nu n}\phi_{\nu n}(r'/c) = \int_{T_r} \phi_{\nu n}(r)J_\nu(rr')(rr')^{1/2}\, dr. \tag{13.3}$$

14. Relationship of the FFST to Slepian functions. Slepian [4] has studied extensively the solutions of the integral equation:

$$\mu_{\nu n}\phi_{\nu n}(r') = \int_0^1 \phi_{\nu n}(r)J_\nu(crr')(crr')^{1/2}\, dr. \tag{14.1}$$

In a previous paper Slepian and Pollak [1] studied the eigenfunctions of (14.1) for $\nu = \pm\frac{1}{2}$. One of the main results found by Slepian was that solutions of (14.1) were also the eigenfunctions of the following Sturm–Liouville equation:

$$\left[\frac{d}{dx}(1 - x^2)\frac{d}{dx} - \left(c^2x^2 + \frac{\nu^2 - \frac{1}{4}}{x^2}\right)\right]\phi_{\nu n} = \mathfrak{X}_{\nu n}\phi_{\nu n}\,. \tag{14.2}$$

This equation is similar to the one for the prolate spherical wave functions $S_{mn}(c, x)$:

$$\left[\frac{d}{dx}(1 - x^2)\frac{d}{dx} - \left(c^2x^2 + \frac{m^2}{1 - x^2}\right)\right]S_{mn} = \mathfrak{X}_{mn}S_{mn}\,. \tag{14.3}$$

This latter lacks the regular singular point of (14.2) at $x = 0$. Slepian called the solutions of (14.2) generalized prolate spheroidal functions. Hurtley [7] has also found that solutions of (14.1) are the same as those of (14.2). He called these functions hyperspheroidal functions. Since (14.2) is not really a generalization of (14.3) it would be proper, in our opinion, to call the solutions of (14.2) Slepian functions. An acceptable notation might be $Sl_{\nu n}$ with the normalization:

$$\int_0^1 Sl_{\nu n}(c, x)Sl_{\nu m}(c, x)\, dx = \delta_{mn}\,.$$

For the special case $\nu = \pm\frac{1}{2}$ we will have

$$Sl_{-1/2,n}(x) = C_{2n}S_{0,2n}(x), \qquad Sl_{1/2,n}(x) = C_{2n+1}S_{0,2n+1}(x)$$

where $S_{0,n}$ is a prolate spheroidal wave function of order 0 and C_n a constant which depends on the normalization of $S_{0,n}$.

To see the relationship between the FFST and spheroidal functions compare the integral equation (2.1) with Eq. (3.22).

We see from Secs. 12 and 13 that when T is an N-dimensional sphere and $\Omega = cT$ the radial part $\hat{\gamma}$ of γ is related to solutions of (14.1) through $\hat{\gamma} = r^\alpha\phi$ where α and the parameter ν in (14.1) depend on N. Morrison [8] has generalized this to N-dimensional ellipses and found the γ's to be separable in ellipsoidal coordinates. He found differential equations for the functions thus obtained.

15. No differential equation for more general cases. It seems that no second-order differential equation exists for the more general cases of (12.9) or (13.2). Morrison [9] mentions that Slepian has indicated to him in a private communication that no second-or fourth-order self-adjoint (Sturm–Liouville) differential operator with polynomial coefficients commutes with the band pass kernel $K(t) = bt^{-1} \sin(at) \cos(bt)$. This corresponds to the case $T = [-1, 1]$, $\Omega = [-b, -a] \cup [a, b]$. Morrison then shows that eigenfunctions of this kernel which are of the same parity and possess a degenerate eigenvalue satisfy a fourth-order differential equation. Slepian and Pollak have indicated [1] that they found degeneracy but not whether these eigenfunctions were of the same parity. Our calculations, to be reported in a subsequent paper, show that there exist degenerate eigenfunctions, but only of opposite parity. Their existence can in fact be shown without computation.

We will show that if T is symmetric about the origin and not a simple interval $[-c, c]$ then there exists no second-order Sturm–Liouville equation for the γ's. If T is a non-symmetric simple interval, then the eigenfunctions are related through the shifting theorem (4.4) to those with a symmetric domain.

Let us assume for simplicity that T is composed of only two intervals, $[-b, -a] \cup [a, b]$, while Ω is arbitrary but bounded. The γ's are band-limited and orthogonal over T. The Sturm–Liouville equation is of the form:

$$(d/dx)p(d/dx)\gamma_i + q\gamma_i = \mathfrak{X}_i\gamma_i\,. \tag{15.1}$$

Multiplying this equation by γ_j, substracting from it the same expression with i and j interchanged and integrating over T yields

$$(\mathfrak{X}_i - \mathfrak{X}_j)(\gamma_i\,, \gamma_j)_T = P(x)|_a^b$$

where $P = 2p(\gamma_i\gamma_j' - \gamma_j\gamma_i')$. Now $P(b) - P(a) = 0$ since the γ's are orthogonal over T. If either $p(a)$ or $p(b)$ is zero, this would imply that $P(a) = P(b) = 0$ and one could use this fact to show that the γ's are orthogonal over $[-a, a]$ and $[-b, b]$. This in turn would imply that the γ's are extrema of

$$||F_\Omega f||^2_{[-a,a]}\, ||f||_\Omega^{-2}, \qquad ||F_\Omega f||^2_{[-b,b]}\, ||f||_\Omega^{-2},$$

and hence that the FFST corresponding to two different domains T were identical. This is clearly not the case. Suppose γ is an eigenfunction corresponding to two different domains, T and T':

$$\lambda_1\gamma(t') = \int_T \gamma(t)K(t - t')\,dt, \tag{15.2}$$

$$\lambda_2\gamma(t') = \int_{T'} \gamma(t)K(t - t')\,dt. \tag{15.3}$$

Multiplying Eq. (15.2) by λ_2 and Eq. (15.3) by λ_1 and subtracting one equation from the other yields

$$\int_{T\cup T'} h(t)K(t - t')\,dt = 0, \tag{15.4}$$

where

$$\begin{aligned} h(t) &= \lambda_2\gamma(t) && t \in T - T' \\ &= (\lambda_2 - \lambda_1)\gamma(t) && t \in T \cap T' \\ &= -\lambda_1\gamma(t) && t \in T' - T. \end{aligned}$$

Rewrite (15.4) in the form

$$\int_{\Omega} e^{-i\omega t'} \int_{T \cup T'} h(t) e^{i\omega t}\, dt\, d\omega = 0. \tag{15.5}$$

This equation can be satisfied for all t' if and only if the integral over t is identically zero for $\omega \in \Omega$. Since $T \cup T'$ is a finite domain the function defined by the integral is analytical and therefore zero everywhere if it is zero in Ω. Since γ itself is analytical the only solution is that $T = T'$ and $\lambda_1 = \lambda_2$. Hence neither $p(a)$ or $p(b)$ is zero. We are left with the infinite set of equations $f_{ij} = 0$ where $f_{ij} = p[\gamma_i\gamma_j' - \gamma_j\gamma_i']_b^a$.

Now let f and g be any two Ω-limited Fourier transforms of some functions F and G in $\mathcal{L}^2$. Both functions can be written as series in terms of γ's: $f = \sum a_m\gamma_m$, $g = \sum b_n\gamma_n$. The series $\sum_{ij} a_ib_jf_{ij}$ will converge to:

$$p(b)[f(b)g'(b) - g(b)f'(b)] - p(a)[f(a)g'(a) + g(a)f'(a)] = 0 \tag{15.6}$$

Since F and G are arbitrary (15.6) must hold for all Ω-limited Fourier transforms and this is not the case. Hence the γ's cannot satisfy a Sturm–Liouville equation. The proof is also valid for more complicated domains T.

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