A NOTE ON OSEEN FLOW*

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It is known that the stream function for Oseen flow past a parabolic cylinder (Stewartson; see Wilkinson [4], Kaplan [3]), and the axially symmetric flow past a paraboloid of revolution can be found in a simple form by employing the Oseen splitting theorem and then utilizing parabolic coordinates. In the case of the parabolic cylinder the limiting solution (Burgers flow) for the semi-infinite flat plate can be determined and this procedure is in fact simpler than the derivation by the Wiener–Hopf technique carried out by Lewis and Carrier [1] and for magnetohydrodynamic flow by Greenspan and Carrier [2]. However, in the case of the paraboloid considered by Wilkinson [4] the limiting solution reduces to a uniform stream. The fluid does not recognize the presence of the needle and a solution for streaming flow past a semi-infinite needle in which the fluid velocity vanishes on the needle and the vorticity is nonzero apparently does not exist. The object of the present note is to show that such a flow can be constructed and is derived quite simply as a fractional integral of the corresponding two-dimensional flow. The integral operator, in general, maps two-dimensional potential, Stokes and Oseen flows into axially symmetric flows in three dimensions, and has been discussed in [5]. In particular, it is well suited to boundaries which occupy part of the axis or for thin discs of finite radius. Boundary-value problems of the type considered in this paper were first solved by Weinstein [6] for the generalized Tricomi equation by finding fundamental solutions in the elliptic halfplane which can be analytically continued into the hyperbolic halfplane. One of the main results is that the fluid velocity components and vorticity on the axis are, apart from constant multiples, identical to the corresponding two-dimensional quantities on the axis. The vorticity is singular at the tip of the needle, decays algebraically downstream on the axis and decays exponentially upstream. In view of the fact that axially symmetric problems converge more rapidly than two-dimensional flows and in view, also, of the central position of the flat plate problem in steady two-dimensional incompressible viscous flow, the following solution for streaming flow past a semi-infinite needle would seem to be of some interest. In particular, the axially symmetric boundary-layer equations for a needle would appear to be a problem equally as interesting; this, however, is not considered in the present short note. Finally, it is noted that the method of solution given here is applicable to streaming flow of a conducting fluid in the presence of an aligned field past a semi-infinite needle.

Let (x, ρ) be cylindrical polar coordinates. Then the Stokes stream function $\psi_3(x, \rho)$ for axially symmetric flow is defined by

$$u_3 = -(1/\rho)(\partial\psi_3/\partial\rho), \qquad v_3 = (1/\rho)(\partial\psi_3/\partial x) \tag{1}$$

where $\mathbf{q}_3 = u_3\hat{k} + v_3\hat{\rho}$ is the fluid velocity vector. The linearized nondimensional Oseen

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equation for axially symmetric viscous flow is given by

$$L_{-1}\left(L_{-1} + \frac{\partial}{\partial x}\right)\psi_3 = 0 \tag{2}$$

where the operator L_k is defined by

$$L_k \equiv (\partial^2/\partial x^2) + (\partial^2/\partial \rho^2) + (k/\rho)(\partial/\partial \rho), \tag{3}$$

which is the axially symmetric Laplacian in a space of $k + 2$ dimensions. Consider now the integral operator

$$\psi_3(x, \rho) = \int_0^\rho \frac{\psi_2(x, y)y\, dy}{(\rho^2 - y^2)^{1/2}} \tag{4}$$

where $\psi_2(x, y)$ may be interpreted as the Earnshaw stream function for two-dimensional flow in the (x, y) plane and is such that the axis is a streamline; that is, $\psi_2(x, 0) = 0$. There is a simple inverse transformation of (4) given by

$$\psi_2(x, y) = \frac{2}{\pi y}\frac{\partial}{\partial y}\int_0^y \frac{\psi_3(x, \rho)\rho\, d\rho}{(y^2 - \rho^2)^{1/2}}, \tag{5}$$

so that there is a one-to-one correspondence between the axially symmetric and two-dimensional flow. Now if the two-dimensional flow fluid velocity is $\mathbf{q}_2 = u_2(x, y)\hat{\imath} + v_2(x, y)\hat{\jmath}$ then it is readily verified that

$$u_3(x, \rho) = \int_0^\rho \frac{u_2(x, y)\, dy}{(\rho^2 - y^2)^{1/2}}, \qquad u_2(x, y) = \frac{2}{\pi}\frac{\partial}{\partial y}\int_0^y \frac{u_3(x, \rho)\rho\, d\rho}{(y^2 - \rho^2)^{1/2}}, \tag{6a}$$

$$v_3(x, \rho) = \frac{1}{\rho}\int_0^\rho \frac{v_2(x, y)y\, dy}{(\rho^2 - y^2)^{1/2}}, \qquad v_2(x, y) = \frac{2}{\pi y}\frac{\partial}{\partial y}\int_0^y \frac{v_3(x, \rho)\rho\, d\rho}{(y^2 - \rho^2)^{1/2}} \tag{6b}$$

and

$$L_{-1}(\psi_3) = \int_0^\rho \frac{L_0(\psi_2)y\, dy}{(\rho^2 - y^2)^{1/2}}, \qquad L_0(\psi_2) = \frac{2}{\pi y}\frac{\partial}{\partial y}\int_0^y \frac{L_{-1}(\psi_2)\rho\, d\rho}{(y^2 - \rho^2)^{1/2}}, \tag{6c}$$

where u_2, v_2 are prescribed in terms of the Earnshaw stream function ψ_2 by the formulae

$$u_2 = -(\partial\psi_2/\partial y), \qquad v_2 = (\partial\psi_2/\partial x). \tag{7}$$

In particular, it follows that

$$u_3(x, 0) = \tfrac{1}{2}\pi u_2(x, 0), \qquad v_3(x, 0) = v_2(x, 0) = 0. \tag{8}$$

Now let ψ_2 be a solution of the two-dimensional Oseen equation

$$L_0(L_0 + (\partial/\partial x))\psi_2 = 0. \tag{9}$$

Then the general solution can be found from the Oseen splitting theorem given by

$$\psi_2 = u + v \tag{10}$$

where u and v are general solutions of

$$L_0(u) = 0, \qquad (L_0 + (\partial/\partial x))v = 0. \tag{11}$$

Again, since

$$\left(L_{-1} + \frac{\partial}{\partial x}\right)\psi_3 = \int_0^\rho \frac{(L_0 + (\partial/\partial x))\psi_2 y\, dy}{(\rho^2 - y^2)^{1/2}}, \tag{12}$$

it follows from (6c) and (12) that if ψ_2 is a solution of (9) then ψ_3 is a solution of (2). The x-axis is a streamline for both flows and if the velocity along the axis is known for one flow then it is known for the other. It will now be shown that a uniform stream in the (x, y) plane maps into a uniform stream for the axially symmetric flow except the magnitude of the speeds are different. Set $u = (2/\pi)y$, $v = 0$; then $\psi_2 = (2/\pi)y$, which is a uniform stream of speed of magnitude $2/\pi$ directed along the negative x-axis, and the corresponding ψ_3 is given by

$$\psi_3 = \frac{2}{\pi}\int_0^\rho \frac{y^2\,dy}{(\rho^2 - y^2)^{1/2}} = \tfrac{1}{2}\rho^2, \tag{13}$$

which is a uniform stream of speed unity directed along the negative x-axis. Now for Oseen flow past a semi-infinite flat plate the boundary-value problem is

$$L_0(L_0 + (\partial/\partial x))\psi_2 = 0,$$

$$\psi_2 = \frac{\partial \psi_2}{\partial y} = 0, \qquad y = 0, x < 0, \qquad \psi_2 \sim \frac{2}{\pi} y \quad \text{as} \quad x^2 + y^2 \to \infty, \tag{14}$$

with ψ_2 odd in y. This boundary-value problem maps into the axially symmetric boundary-value problem defined as follows:

$$L_{-1}\left(L_{-1} + \frac{\partial}{\partial x}\right)\psi_3 = 0 \tag{15}$$

with $\psi_3 = (\partial\psi_3/\partial\rho) = 0$, $\rho = 0$, $x < 0$, $\psi_3 \sim \frac{1}{2}\rho^2$ as $\rho^2 + x^2 \to \infty$. To solve the boundary value problem posed by (14) the simplest procedure is to introduce parabolic coordinates (ξ, η) defined by

$$x + iy = (\xi + i\eta)^2; \tag{16}$$

then the equations satisfied by u and v are

$$u_{\xi\xi} + u_{\eta\eta} = 0, \qquad v_{\xi\xi} + v_{\eta\eta} + 2\xi v_\xi - 2\eta v_\eta = 0, \tag{17}$$

and the complete solution is found to be

$$\psi_2 = (4\eta/\pi)\{\xi + (1/\sqrt{\pi})[\exp(-\xi^2) - 1 - 2\xi \operatorname{erfc} \xi]\}. \tag{18}$$

The stream function for axially symmetric flow is then found from (4). The vorticity of the two-dimensional flow is given by (6c),

$$\operatorname{curl} \mathbf{q}_3 = \zeta_3 \hat{\phi} \tag{19}$$

where $\zeta_3 = L_{-1}(\psi_3)/\rho$ and is related to the vorticity of the two-dimensional flow by (6c), that is

$$\zeta_3(x, \rho) = \frac{1}{\rho}\int_0^\rho \frac{\zeta_2(x, y)y\,dy}{(\rho^2 - y^2)^{1/2}}, \qquad \zeta_2 = L_0(\psi_2). \tag{20}$$

Now from (18)

$$\zeta_2 = \frac{\eta \exp(-\xi^2)}{\pi^{3/2}(\xi^2 + \eta^2)}, \tag{21}$$

and on the axis $\zeta_3(x, 0) = \zeta_2(x, 0)$, so that since $\xi = 0$ corresponds to $y = 0$, $x < 0$,

$\eta = (-x)^{1/2}$ and $\eta = 0$ corresponds to $y = 0$, $x > 0$, $\xi = x^{1/2}$, it follows that

$$\begin{aligned} \zeta_3(x, 0) &= \frac{1}{\pi^{3/2}} \frac{1}{(-x)^{1/2}}, \qquad x < 0, \\ &= 0, \qquad x > 0. \end{aligned} \tag{22}$$

Thus the vorticity on the needle is identical to the vorticity on the flat plate. Although the vorticity is finite on the needle except at the tip, the quantity ring vorticity $l = L_{-1}(\psi_3)/\rho^2$ is infinite along the negative x-axis. This quantity is usually finite on axially symmetric bodies. However, the velocity and its derivatives are finite everywhere except on the tip of the needle. This is readily checked from (6a), (6b), (6c) and the results

$$\begin{aligned} \frac{\partial}{\partial x} [L_{-1}(\psi_3)] &= \int_0^\rho \frac{(\partial/\partial x)\{L_0(\psi_2)\} y \, dy}{(\rho^2 - y^2)^{1/2}}, \\ \frac{\partial}{\partial \rho} [L_{-1}(\psi_3)] &= \rho \int_0^\rho \frac{(\partial/\partial y)(L_0(\psi_2)) \, dy}{(\rho^2 - y^2)^{1/2}} + L_0(\psi_2)|_{y=0}. \end{aligned} \tag{23}$$

It is of interest to mention briefly that the flow past a paraboloid of revolution can be obtained simply by the introduction of parabolic coordinates defined by $x + i\rho = (\xi + i\eta)^2$. The stream function is determined by the same method as before and is given by

$$\psi_3 = 2\eta^2 \left\{ \xi^2 - \xi_0^2 + \frac{2}{\displaystyle\int_{\xi_0}^\infty \frac{e^{-u^2} \, du}{u^3}} \left[\xi^2 \int_\xi^\infty \frac{e^{-u^2} \, du}{u^3} - \xi_0^2 \int_{\xi_0}^\infty \frac{e^{-u^2} \, du}{u^3} \right] \right\}, \tag{24}$$

where $\xi \geq \xi_0$ corresponds to the exterior of the paraboloid. Now as $\xi_0 \to 0$, $\psi_3 \to 2\xi^2\eta^2 = \frac{1}{2}\rho^2$, which is a uniform stream. In this limiting situation the fluid does not recognize the presence of the needle since the vorticity is zero on the axis and the no-slip condition on the negative x-axis is not satisfied. The flow locally at the leading edge of the two-dimensional motion, in polar coordinates defined by $x = r \cos \theta$, $y = r \sin \theta$, is given by:

$$\psi_2 \sim \frac{4}{\pi^{3/2}} y\xi = 2\left(\frac{r}{\pi}\right)^{3/2} \left(\sin \frac{3\theta}{2} + \sin \frac{\theta}{2}\right), \tag{25}$$

which is clearly Stokes flow since the right-hand side of (23) is a two-dimensional biharmonic function. Since two-dimensional biharmonic functions are mapped into solutions of $L_{-1}^2(\psi) = 0$, the flow locally at the tip of the needle must be Stokes flow and is given by

$$\psi_3 \sim \frac{2^{1/2} 4 r^{5/2} \sin^2 \theta}{\pi^{3/2}} \left\{ \int_0^{1/2\pi} [\cos \theta + (\sin^2 \theta \sin^2 \lambda + \cos^2 \theta)^{1/2}]^{1/2} \sin^2 \lambda \, d\lambda \right\}. \tag{26}$$

where in this case $x = r \cos \theta$, $\rho = r \sin \theta$. Also the final form for ψ_3 expressed in (x, ρ) coordinates is, from (4) and (18), given by

$$\begin{aligned} \psi_3 = \tfrac{1}{2}\rho^2 &+ \left(\frac{2}{\pi}\right)^{3/2} \int_0^\rho \frac{y^2(\exp(-(r + x)/2) - 1) \, dy}{(r + x)^{1/2}(\rho^2 - y^2)^{1/2}} \\ &- \frac{4}{\pi^{3/2}} \int_0^\rho \frac{y^2 \operatorname{erfc} [(r + x)/2]^{1/2} \, dy}{(\rho^2 - y^2)^{1/2}} \end{aligned} \tag{27}$$

where $r = (x^2 + y^2)^{1/2}$.

For a more complete proof when there is a distribution of singularity on the axis, see Mathematics Research Center T-S R1251, University of Wisconsin, 1972.

Appendix. In this appendix the basic identity required in the note will be proved. Set

$$w(x, \rho) = \int_0^\rho \frac{v(x, y)y\,dy}{(\rho^2 - y^2)^{1/2}}$$

where $v(x, 0) = 0$, and write $s = \rho^2$, $t = y^2$. A simple computation gives

$$L_{-1} \equiv 4s\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial x^2}, \qquad L_0 = 4t\frac{\partial^2}{\partial t^2} + 2\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}, \qquad W = \frac{1}{2}\int_0^s \frac{v\,dt}{(s - t)^{1/2}}.$$

Now

$$\frac{\partial^2 W}{\partial x^2} = \frac{1}{2}\int_0^s \frac{\partial^2 v}{\partial x^2}\frac{dt}{(s - t)^{1/2}}$$

and

$$\frac{\partial W}{\partial s} = \frac{1}{2}\int_0^s \frac{\partial v}{\partial t}\frac{dt}{(s - t)^{1/2}},$$

since $v|_{t=0} = 0$. Again,

$$\frac{\partial^2 W}{\partial s^2} = \frac{1}{2}\int_0^s \frac{\partial^2 v}{\partial t^2}\frac{dt}{(s - t)^{1/2}} + \frac{1}{2s^{1/2}}\frac{\partial v}{\partial t}\bigg|_{t=0}$$

and

$$\begin{aligned} 4s\frac{\partial^2 W}{\partial s^2} &= 2\int_0^s \frac{(s - t + t)}{(s - t)^{1/2}}\frac{\partial^2 v}{\partial t^2}\,dt + 2s^{1/2}\frac{\partial v}{\partial t}\bigg|_{t=0} \\ &= 2\int_0^s (s - t)^{1/2}\frac{\partial^2 v}{\partial t^2}\,dt + 2\int_0^s + t\frac{\partial^2 v}{\partial t^2}\frac{dt}{(s - t)^{1/2}} + 2s^{1/2}\frac{\partial v}{\partial t}\bigg|_{t=0} \\ &= 2\left[(s - t)^{1/2}\frac{\partial v}{\partial t}\right]_0^s + \int_0^s \left(2t\frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t}\right)\frac{dt}{(s - t)^{1/2}} + 2s^{1/2}\frac{\partial v}{\partial t}\bigg|_{t=0} \\ &= \frac{1}{2}\int_0^s \left(4t\frac{\partial^2 v}{\partial t^2} + 2\frac{\partial v}{\partial t}\right)\frac{dt}{(s - t)^{1/2}}. \end{aligned}$$

Thus

$$L_{-1}(W) = \frac{1}{2}\int_0^s \frac{L_0(v)\,dt}{(s - t)^{1/2}} = \int_0^s \frac{L_0(v)y\,dy}{(\rho^2 - y^2)^{1/2}},$$

as required.

Asymptotic formula for the vorticity distribution. The vorticity distribution $\zeta_3(x, \rho)$ is given by

$$\zeta_3 = \frac{1}{\pi^{3/2}2^{1/2}\rho}\int_0^\rho \frac{(r^1 - x)^{1/2}\exp -\left(\dfrac{x + r^1}{2}\right)y\,dy}{r^1(\rho^2 - y^2)^{1/2}}$$

where $r^1 = (x^2 + y^2)^{1/2}$. On making the successive transformations $y = \rho v$, $\rho = r \sin \theta$, $x = r \cos \theta$, $s = (\cos^2 \theta + v^2 \sin^2 \theta)^{1/2}$ and $s = u + |\cos \theta|$, ζ_3 can be expressed as

$$\zeta_3 = \frac{\rho^{-(r/2)(\cos \theta + |\cos \theta|)}}{(2r)^{1/2}\pi^{3/2} \sin \theta} \int_0^{1-|\cos \theta|} \frac{\{u + |\cos \theta| - \cos \theta\}^{1/2}}{[1 - (u + |\cos \theta|)^2]^{1/2}} \exp\left(-\frac{r}{2} u\right) du.$$

By applying Watson's lemma, it is found that for large r

$$\zeta_3 \sim \frac{e^{-r \cos \theta}}{\pi r^2 \sin^2 \theta}, \qquad 0 < \theta < \pi/2$$

$$\sim \frac{1}{\pi r^2}, \qquad \theta = \pi/2$$

$$\sim \frac{2\,|\cos \theta|^{1/2}}{\pi^{3/2} \sin^2 \theta r^{3/2}}, \qquad \pi/2 < \theta < \pi.$$

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