BUCKLING OF STOCHASTICALLY IMPERFECT COLUMNS ON NONLINEAR ELASTIC FOUNDATIONS*

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Abstract. Approximate asymptotic expressions are obtained for the buckling stresses and autocorrelation of the lateral displacement of infinitely long imperfect columns resting on nonlinear elastic foundations. The imperfections are assumed to be homogeneous Gaussian random functions with known autocorrelation. The formulas are discussed and compared with previous results obtained by means of truncated hierarchy and equivalent linearization techniques.

Introduction. In this paper a perturbation scheme is used to study a model imperfection-sensitive structure. We consider an ensemble of infinitely long imperfect columns resting on nonlinear elastic foundations. The stress-free initial displacements of the columns are assumed to be homogeneous zero-mean Gaussian random functions of positions along the column. In the analysis, approximate asymptotic expressions that are applicable for small mean square of the imperfections are sought for the buckling stresses and the autocorrelation of the displacements.

In an earlier study of this problem [1] Fraser and Budiansky used an equivalent linearization technique to obtain the buckling stresses for random imperfections with two-parameter exponential-cosine autocorrelation functions. In a subsequent study [2] in which other types of imperfections were considered, Amazigo, Budiansky and Carrier obtained asymptotic expressions for the buckling load by means of both equivalent linearization and truncated-hierarchy methods. The perturbation scheme used in this paper appears more satisfactory and yields slightly different asymptotic expressions for the buckling load. In addition an asymptotic expression is obtained for the buckling displacement.

Differential equation. The nondimensional form of the differential equation governing the lateral displacement of an infinite column on a "softening" nonlinear elastic foundation is

$$Lw - w^3 = -2\lambda\epsilon w_0'', \qquad -\infty < x < \infty, \qquad (1)$$

where $L(\ ) = (\ )'''' + 2\lambda(\ )'' + (\ )$ with $(\ )' = (d/dx)(\ )$. The nondimensional axial coordinate x, lateral deflection w, axial load parameter λ, and stress-free initial displacement w_0 are related to the physical quantities by

$$x = (k_1/EI)^{1/4}X, \qquad w = (k_3/k_1)^{1/2}W, \qquad \lambda = P/2(EIk_1)^{1/2}, \qquad (2)$$

$\epsilon w_0 = (k_3/k_1)^{1/2} W_0$, where ϵ is a small imperfection parameter. In these equations EI is the bending stiffness of the column. The lateral displacement $W(X)$ of the column is restrained by an elastic foundation that produces a restoring force per unit length of $k_1 W - k_3 W^3$ with k_1 , $k_3 > 0$. We consider an ensemble of such columns, with each column subjected to the same axial load P. The initial stress-free displacement (imperfection) $W_0(X)$ is assumed to be a homogeneous zero-mean Gaussian random function of position X along the column.

The load parameter λ is defined so that the lowest eigenvalue of the linear problem $Lw = 0$ with $w_0 \equiv 0$ is $\lambda = 1$. The corresponding eigenfunction is

$$w = \cos (x - \theta)$$

where θ is an arbitrary phase angle.

For the nonlinear, nonhomogeneous equation (1), we seek a relation between the load parameter λ, the imperfection parameter ϵ, and the mean square Δ^2 of the displacement w. The buckling load of the structure $\bar{\lambda}$ is the maximum value of λ for the branch of the solutions satisfying the condition $\lambda = 0$ for $\Delta = 0$. We seek an asymptotic expression for $\bar{\lambda}$ applicable for sufficiently small values of ϵ and Δ.

Formulation of perturbation scheme. The imperfection function $w_0(x)$ is a sample function from an ensemble of zero-mean, homogeneous Gaussian random functions with known autocorrelation function $R_{00}(\zeta)$. Thus

$$\langle w_0(x) \rangle = 0, \qquad \langle w_0(x + \zeta) w_0(x) \rangle = R_{00}(\zeta), \tag{3}$$

where the angular bracket $\langle \cdots \rangle$ denotes ensemble average. The power spectral density of w_0 is defined by

$$S_{00}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{00}(\zeta) e^{-i\omega\zeta} \, d\zeta. \tag{4}$$[1]

We consider λ to be prescribed and to satisfy the inequality $0 < \lambda < 1$, and expand w in the form

$$w(x) = \sum_{i=1}^{\infty} \epsilon^i w_i(x). \tag{5}$$

Substituting (5) into the differential equation (1) and equating coefficients of successive powers of ϵ gives

$$Lw_1 = -2\lambda w_0'', \qquad Lw_2 = 0, \qquad Lw_3 = w_1^3 ,$$
$$Lw_4 = 3w_1^2 w_2 , \qquad Lw_5 = 3w_1 w_2^2 + 3w_1^2 w_3 , \quad \text{etc.} \tag{6}$$

We observe that for $\lambda < 1$, the bounded analytic solution of $Lw_2 = 0$ is $w_2 \equiv 0$. Consequently $w_n \equiv 0$ for n even. Eqs. (5) and (6) are then equivalent to

$$w(x) = \sum_{i=1}^{\infty} \epsilon^{2i-1} w_{2i-1}(x), \tag{7}$$

$$Lw_1 = -2\lambda w_0'', \tag{8}$$

$$Lw_3 = w_1^3 , \tag{9}$$

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[1] Unless otherwise specified the limits of all integrals are $-\infty$, ∞.

$$Lw_5 = 3w_1^2 w_3 , \quad \text{etc.} \tag{10}$$

Let

$$\Delta_{ij} \equiv \langle w_i(x) w_j(x) \rangle, \tag{11}$$

$$R_{11}(\zeta) = \langle w_1(x + \zeta) w_1(x) \rangle, \tag{12}$$

and Δ^2 be the mean square of the deflection w, that is,

$$\Delta^2 \equiv \langle w^2(x) \rangle. \tag{13}$$

Substituting (7) into (13) and using (11) gives

$$\Delta^2 = \epsilon^2 \Delta_{11} + 2\epsilon^4 \Delta_{13} + \epsilon^6 (2\Delta_{15} + \Delta_{33}) + 0(\epsilon^8). \tag{14}$$

Since the Δ_{ij}'s are functions of λ, Eq. (14) gives a relation between Δ^2, λ, ϵ. In principle, then, the buckling load should be obtained from (14) by setting $d\lambda/d\Delta^2 = 0$. This scheme fails because Δ^2 is a multivalued function of λ and the series does not converge for $\Delta^2 > \bar{\Delta}^2$ (the critical mean square).

This difficulty is overcome if we reverse the series (14) and seek the coefficients α_i such that

$$\epsilon^2 = \alpha_1 \Delta^2 + \alpha_2 \Delta^4 + \alpha_3 \Delta^6 + 0(\Delta^8). \tag{15}$$

The reverse series (15) gives $\bar{\Delta}^2$ as an analytic function of ϵ^2, and it uniquely defines $\bar{\Delta}$ as zero for $\epsilon = 0$. Note that $\bar{\lambda} = 1$ for $\epsilon = 0$.

The coefficients α_i may be obtained by using Lagrange's formula for the reversion of series (see, for example, [3]). Alternatively, substituting (15) into (14) and equating the powers of Δ^2 gives

$$\alpha_1 = 1/\Delta_{11} , \tag{16}$$

$$\alpha_2 = -2\Delta_{13}/\Delta_{11}^3 , \tag{17}$$

$$\alpha_3 = 8\Delta_{13}^2/\Delta_{11}^5 - (2\Delta_{15} + \Delta_{33})/\Delta_{11}^4 . \tag{18}$$

We now consider a two-term approximation which is obtained by retention of only the first two terms on the right of Eq. (15). Considerations of a higher-order approximation would tend to obscure the basic ideas of the perturbation scheme.

Two-term approximation. Retaining two terms in the expansion (15) gives

$$\epsilon^2 \approx \alpha_1(\lambda) \Delta^2 + \alpha_2(\lambda) \Delta^4. \tag{19}$$

Maximizing λ with respect to Δ^2 and using (16), (17) gives

$$8(\Delta_{13}(\bar{\lambda})/\Delta_{11}(\bar{\lambda}))\epsilon^2 = 1 \tag{20}$$

as an approximate relation between the buckling load $\bar{\lambda}$ and the imperfection parameter ϵ. We now seek expressions for $\Delta_{11}(\lambda)$, and $\Delta_{13}(\lambda)$.

The solution of Eq. (8) is

$$w_1(x) = -2\lambda \int G(x - y) w_0''(y) \, dy \tag{21}$$

where

$$G(\zeta) = \tfrac{1}{4} e^{-b|\zeta|} \left\{ \frac{\cos a\zeta}{b} - \frac{\sin a |\zeta|}{a} \right\}, \tag{22}$$

$$a = [\tfrac{1}{2}(1 + \lambda)]^{1/2}, \qquad b = [\tfrac{1}{2}(1 - \lambda)]^{1/2},$$

and the Fourier transform $Q(\omega)$ of $G(\zeta)$ is

$$Q(\omega) = \int G(\zeta)e^{i\omega\zeta}\,d\zeta = (\omega^4 - 2\lambda\omega^2 + 1)^{-1}. \tag{23}$$

Taking ensemble average of (21) gives $\langle w_1(x)\rangle = 0$ since $\langle w_0(x)\rangle = 0$. As in [1], we find that the spectral density $S_{11}(\omega)$ of w_1 is

$$S_{11}(\omega) = 4\lambda^2\omega^4 Q^2(\omega)S_{00}(\omega).$$

Thus

$$\Delta_{11} = \int S_{11}(\omega)\,d\omega = 4\lambda^2 \int \omega^4 Q^2(\omega)S_{00}(\omega)\,d\omega \tag{24}$$

and Fourier transform R_{11} of S_{11} is

$$R_{11}(\zeta) = 4\lambda^2 \int \omega^4 Q^2(\omega)S_{00}(\omega)e^{i\omega\zeta}\,d\omega. \tag{25}$$

The solution of Eq. (9) is

$$w_3(x) = \int G(x - y)w_1^3(y)\,dy. \tag{26}$$

Now since w_1 is a linear function of the Gaussian random function w_0, it is also Gaussian and hence (see, for example, [4])

$$\langle w_1(x)w_1^3(y)\rangle = 3\Delta_{11} R_{11}(x - y). \tag{27}$$

We observe that w_3 is not Gaussian and hence w is not Gaussian. Thus, multiplying (26) by $w_1(x)$, taking ensemble average and using (27), (23), and (25) gives

$$\Delta_{13} = 12\lambda^2\Delta_{11} \int \omega^4 Q^3(\omega)S_{00}(\omega)\,d\omega. \tag{28}$$

Since Eq. (20) for the buckling load $\bar{\lambda}$ depends only on the ratio Δ_{13}/Δ_{11} with $\lambda = \bar{\lambda}$, we need to evaluate the integral on the right of (28) for any given autocorrelation functions $R_{00}(\zeta)$.

We seek, however, an asymptotic expression for Δ_{13}/Δ_{11} valid for $\lambda \to 1$. Consider the integral

$$A_n \equiv \int \omega^4 [Q(\omega)]^n S_{00}(\omega)\,d\omega, \qquad n > 1 \tag{29}$$

and, noting that the integrand is an even function of ω, we have

$$A_n = 2\int_0^\infty \omega^4 [Q(\omega)]^n S_{00}(\omega)\,d\omega.$$

Let

$$\lambda = 1 - \tfrac{1}{2}\delta^2 \tag{30}$$

and make a change of variable $\omega^2 = 1 + \delta\rho$; then

$$A_n = \frac{S_{00}(1)}{\delta^{2n-1}} \left\{ \int \frac{d\rho}{(\rho^2 + 1)^n} + O(\delta) \right\}.$$

Integrating, using calculus of residues, gives

$$A_n = \frac{\pi S_{00}(1)}{\delta^{2n-1}} \frac{(n-1)(2n-3)!}{2^{2n-3}[(n-1)!]^2} [1 + O(\delta)]. \tag{31}$$

Substituting for δ^2 and using (31) and (28) in (20) for $\lambda = \bar{\lambda}$ gives the approximate asymptotic expression for $\bar{\lambda}$

$$(1 - \bar{\lambda})^{5/4} \approx 3(2)^{-1/4}[\pi S_{00}(1)]^{1/2} \bar{\lambda} \epsilon. \tag{32}$$

As in [2], we observe that asymptotically the buckling load depends only on the value of the spectral density at $\omega = 1$. Eq. (32) gives a more conservative estimate of the buckling load $\bar{\lambda}$ than the result of [2], namely

$$(1 - \bar{\lambda})^{5/4} \approx \tfrac{5}{2}(\tfrac{5}{6})^{1/4}[\pi S_{00}(1)]^{1/2} \bar{\lambda} \epsilon. \tag{33}$$

These formulas are compared in Fig. 1, where $\bar{\lambda}$ is plotted against the imperfection measure $[\pi S_{00}(1)]^{1/2} \epsilon$.

Asymptotic expression for the autocorrelation. We seek an asymptotic expression for the autocorrelation $R(\zeta)$ of the deflection $w(x)$ for the case when $R_{00}(\zeta)$ decays exponentially. Assume that

$$|R_{00}(\zeta)| < M e^{-\alpha \zeta}, \qquad (M, \alpha > 0). \tag{34}$$

Substituting (7) into the definition of $R(\zeta)$, namely $R(\zeta) = \langle w(x + \zeta)w(x) \rangle$ gives

$$R(\zeta) = R_{11}(\zeta)\epsilon^2 + [\langle w_1(x + \zeta)w_3(x) \rangle + \langle w_1(x)w_3(x + \zeta) \rangle]\epsilon^4 + O(\epsilon^6).$$

Now we use the integral expression (26) for w_3 and properties of Fourier transforms to obtain

$$R(\zeta) = \epsilon^2 \cdot 4\lambda^2 \int \omega^4[Q(\omega)]^2 S_{00}(\omega)e^{i\omega\zeta} \, d\omega$$
$$+ \epsilon^4 \cdot 24\lambda^2\Delta_{11} \int \omega^4[Q(\omega)]^3 S_{00}(\omega)e^{-i\omega\zeta} \, d\omega + O(\epsilon^6). \tag{35}$$

Consider the first integral on the right of (35), namely

$$R_{11}(\zeta) = 4\lambda^2 \int \omega^4[Q(\omega)]^2 S_{00}(\omega)e^{i\omega\zeta} \, d\omega. \tag{36}$$

A similar integral was evaluated asymptotically in [2] and [5]. Noting that $S_{00}(\omega)$ is analytic for $|\text{Im } w| < \alpha$ as a consequence of (34), we shift the path of integration below the poles of the integrand for $\zeta > 0$. The poles of $Q(\omega)$ are $\omega_{1,2} = \pm[(1 + \lambda)/2]^{1/2} - i[(1 - \lambda)/2]^{1/2}$. The integral along the new contour is $O(e^{-\beta\zeta})$ where $[(1 - \lambda)/2]^{1/2} < \beta < \alpha$. Hence

$$R_{11}(\zeta) = -8\pi\lambda^2 i \, [\text{Residues of integrand at } \omega_1, \omega_2] + O(e^{-\beta\zeta}) \tag{37}$$

The case $\zeta < 0$ is handled in a similar manner and combining these results gives

$$R_{11}(\zeta) = \frac{\pi\lambda^2 S_{00}(1)}{2^{1/2}(1-\lambda)^{3/2}} \exp\left\{-[(1-\lambda)/2]^{1/2}\,|\zeta|\right\}\cos\zeta + O((1-\lambda)^{-1}) \quad \text{for} \quad \lambda \to 1.$$
$$\text{(38)}$$

Evaluating the second integral in (35) gives

$$\int \omega^4 [Q(\omega)]^3 S_{00}(\omega) e^{-i\omega\zeta}\,d\omega = 3\pi\lambda^2 S_{00}(1)/2^{11/2}(1-\lambda)^{5/2};$$

$$\exp\left\{-[(1-\lambda)/2]^{1/2}\,|\zeta|\right\}\cos\zeta + O((1-\lambda)^{-2}). \qquad (39)$$

Since $\Delta_{11} = R_{11}(0)$, the autocorrelation of the buckling deflection is obtained by using (39), (38), and (37) in (35) and setting $\lambda = \bar{\lambda}$. Thus

$$R(\zeta) \approx \left[\epsilon^2 + \frac{9\pi\bar{\lambda}^2 S_{00}(1)}{2^{5/2}(1-\bar{\lambda})^{5/2}}\,\epsilon^4\right]\frac{\pi\bar{\lambda}^2 S_{00}(1)}{2^{1/2}(1-\bar{\lambda})^{3/2}}\exp\left[-(1-\lambda)/2\right]^{1/2}\,|\zeta|\cos\zeta. \quad (40)$$

Now $\bar{\lambda}$ and ϵ are related by Eq. (32), hence the two terms in the square brackets in (40) are of the same order. Substituting for ϵ in (40) gives
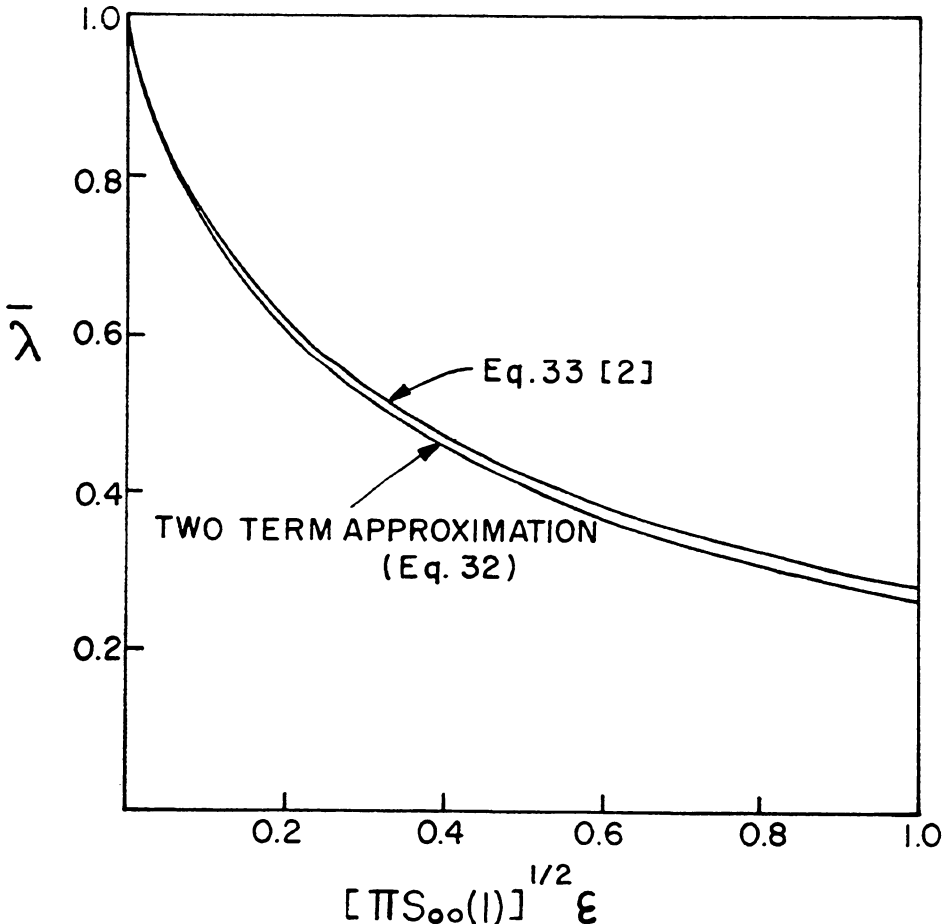

FIG. 1.   Dependence of buckling load on imperfection.

$$R(\zeta) \approx \frac{5}{36} (1 - \bar{\lambda}) \exp \left\{ -[(1 - \bar{\lambda})/2]^{1/2} |\zeta| \right\} \cos \zeta. \tag{41}$$

Concluding remarks.   In comparing the result of this analysis (32) with the result (33) of [2] obtained by means of truncated hierarchy and equivalent linearization, we note that the nature of the dependence of $\bar{\lambda}$ on ϵ and $S_{00}(\omega)$ is the same. The only difference is in the coefficients of $[\pi S_{00}(1)]^{1/2} \bar{\lambda} \epsilon$. This difference (2.52 for (32) and 2.39 for (33)) appears very small, as seen in Fig. 1. An analysis not reported here shows that a three-term approximation does not significantly change the coefficient in (32). This perturbation scheme has also been used to duplicate the results in [2] for modal imperfection. However, three terms were needed in the approximation.

We observe also that this analysis clearly indicates that w is not Gaussian. In the method of equivalent linearization w becomes Gaussian as a consequence of the linearization. For the truncated hierarchy technique w was assumed Gaussian to facilitate truncation of the resulting infinite sequence of equations.

It is hoped that the perturbation scheme developed here can be extended to solve problems of buckling of thin shells with random imperfections.

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