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THE INFLUENCE OF DYNAMICAL THERMAL EXPANSION ON THE PROPAGATION OF PLANE ELASTIC-PLASTIC STRESS WAVES*

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Summary. In this paper the implications of the classical heat conduction equation for the problem of the propagation of plane waves caused by mechanical impulse and sudden heating at the boundary of an elastic-plastic half-space are presented. It is shown that the effect of dynamical thermal expansion is to reduce the jump in the stress at waves of strong discontinuity. The stress and temperature fields dealt with here are assumed to be thermodynamically uncoupled.

1. Introduction. The purpose of this paper is to give a theoretical basis for investigating the effects of dynamical thermal expansion on the propagation of the plane wave caused by simultaneous mechanical impulse and sudden heating at the boundary of an elastic-plastic half-space (Fig. 1).

Investigations concerning the propagation of elastic-plastic waves have been devoted either to the case of pure mechanical impulse [1]–[3] or to the case of pure thermal load. In the latter cases the boundary temperature of the half-space [4]–[6] or rod [7]–[8] were prescribed functions (step-function, linear function) of time. Fine and Kraus [9] have analysed the stress field in an elastic-plastic half-space assuming constant heat flux. It is difficult, however, to agree with the results obtained in their paper in which the existence of a strong discontinuity plastic wave was postulated although the strain at the boundary was continuous. Consequently a negative value was obtained for the jump in plastic work at the front of this wave. This is clearly incorrect and shows that the assumed wave of strong discontinuity is incorrect.

It is not the purpose of this paper to provide a final solution for any given temperature field, but rather to consider generally the character of wave propagation for the class of temperature fields and mechanical impulses shown in Fig. 2. We discuss here only strong mechanical impulses: those which, though reduced by thermal stress at the boundary of the half-space, considerably exceed the yield limit.

The final wave diagram in the t-x^1 plane for the impact mechanical load is obtained as a limiting case of the wave diagram for an applied pressure $p_1(t)$ (dotted line in Fig. 2) which is initially a linear function of time. We assume here that the material constants are independent of the temperature. The temperature is taken into account only in the relation between mean pressure and the relative volume change. The Prandtl–Reuss relations with Mises yield condition are used [10, 11] to describe the material behaviour in the plastic range. In the relatively simple case of one-dimensional strain considered here the resulting equations can be integrated.

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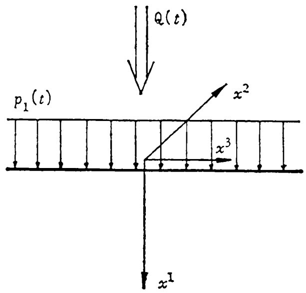

FIG. 1

2. The primary equations. Due to uniform temperature and pressure distribution at the plane $x^1 = 0$ the motion is plane and irrotational (Fig. 1):

$$\epsilon_{ik} = 0 \quad \text{for} \quad i \neq 1, \qquad k \neq 1.$$

From this, and from the stress-strain relations, the following symmetry properties of the stress state can be obtained:

$$\sigma_{22} = \sigma_{33} \quad \text{and} \quad \sigma_{ik} = 0 \quad \text{for} \quad i \neq k.$$

The functions ϵ_{11}, σ_{11}, σ_{22}, σ_{33} depend on x^1 and t only.

The function F,

$$F = \sigma_{11} + 3K\alpha\theta, \tag{1}$$

and the following dimensionless quantities are now introduced:

$$\begin{aligned} &\mathfrak{F} = 2F/3K\alpha T_0\,, \quad e_1 = 2\rho a_0^2\epsilon_{11}/3K\alpha T_0\,, \quad y = a_0x^1/k, \quad \tau = a_0^2t/k, \\ &T = \theta/T_0\,, \quad \mathfrak{F}_s = e_s = \rho a_0^2 Y/3K\alpha T_0\mu, \quad \gamma = K/a_0^2\rho, \\ &S_1 = 2\sigma_{11}/3K\alpha T_0\,, \quad S_2 = 2\sigma_{22}/3K\alpha T_0\,, \quad U = 2\rho a_0^3u/3Kk\alpha T_0\,, \\ &\partial U/\partial y = e_1\,, \quad \partial U/\partial\tau = V = 2\rho a_0(\partial u/\partial t)/3K\alpha T_0\,, \end{aligned} \tag{2}$$

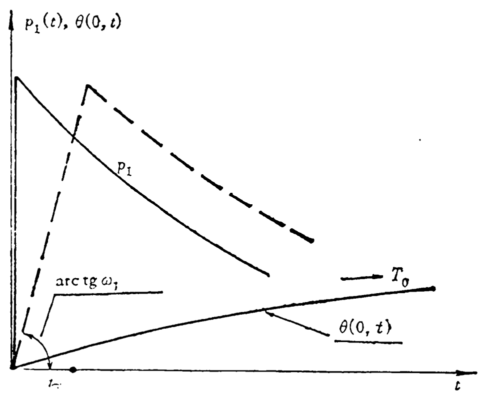

FIG. 2

where $a_0^2 = K + 4\mu/3$, θ is temperature, T_0 is a constant which possesses the dimension of temperature, $K\mu$ are bulk and shear modulus, k is thermal diffusivity, α is thermal expansion, ρ is density, Y is yield stress in tension, u is displacement in the x^1 direction and a_0 is velocity of propagation of the longitudinal elastic waves.

Assuming a zero initial state,

$$e_1 = V = T = 0 \quad \text{for} \quad \tau = 0, \tag{3}$$

and integrating the Prandtl–Reuss equations (taking $\sigma_{ii}/3K = \epsilon_{ii} - 3\alpha\theta$ as the relation between the mean pressure and relative volume change) written for the case considered, we obtain the following relations between stress, strain and temperature:

$$S_1 = \mathfrak{F} - 2T \quad \text{in all regions} \tag{4}$$

$$\begin{aligned} S_2 + 2T &= (3\gamma^2 - 1)\mathfrak{F}/2 \quad \text{in the elastic regions } (|\mathfrak{F}| \le e_s) \\ &= \mathfrak{F} - \tfrac{3}{2}(1 - \gamma^2)(\operatorname{sgn} \mathfrak{F})e_s \quad \text{in the region of primary elastic-plastic state} \\ &\qquad\qquad (|\mathfrak{F}| \ge e_s\,,\ \dot{\mathfrak{F}} \operatorname{sgn} \mathfrak{F} \ge 0) \\ &= (3\gamma^2 - 1)\mathfrak{F}/2 + 3(1 - \gamma^2)\mathfrak{F}_0/2 - 3(1 - \gamma^2)(\operatorname{sgn} \mathfrak{F}_0)e_s/2 \\ &\qquad\qquad \text{in the unloading region } (|\mathfrak{F}| \le |\mathfrak{F}_0|), \end{aligned} \tag{5}$$

where $\mathfrak{F}_0$ is the value of the function $\mathfrak{F}$ when unloading begins (Fig. 3) and the partial time derivative is denoted by a dot. The strain e_1 is the function of $\mathfrak{F}$ shown in Fig. 3. The condition of a nonnegative power of plastic deformation in the zones of elastic-plastic strain is equivalent to $(\operatorname{sgn} \mathfrak{F})\dot{\mathfrak{F}} \ge 0$, and unloading begins when this condition is violated. The equations of motion for the dimensionless quantities take the form

$$\frac{\partial V}{\partial y} = \frac{1}{\gamma_1^2}\frac{\partial \mathfrak{F}}{\partial \tau}, \qquad \frac{\partial V}{\partial \tau} = \frac{\partial \mathfrak{F}}{\partial y} - 2\frac{\partial T}{\partial y}, \tag{6}$$

where

$$\begin{aligned} \gamma_1 &= 1 \quad \text{in the elastic and unloading regions,} \\ &= \gamma \quad \text{in the elastic-plastic regions.} \end{aligned}$$

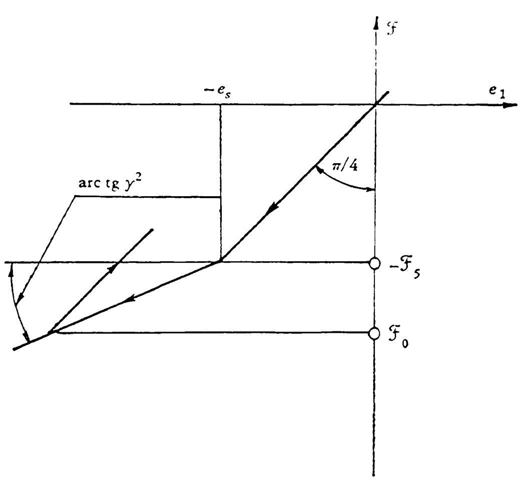

Fig. 3

The following assumptions about temperature field are now accepted:

(a) $T(y, \tau)$ satisfies the homogeneous heat conduction equation

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial T}{\partial \tau}; \tag{7}$$

(b) $T|_{y=0}$ is positive, continuous and increasing from 0 to the value 1 in the time period $0 \leq \tau < \infty$;

(c) $\dot{T}|_{y=0}$ is everywhere bounded.

The set of hyperbolic equations (6) possesses the following particular integrals [12]:

$$\begin{aligned} \mathfrak{F}_1(y, \tau, \gamma_1) &= -2\gamma_1^2 \int_0^\tau \exp\,[\gamma_1^2(\tau - \eta)]T(y, \eta)\, d\eta, \\ V_1(y, \tau, \gamma_1) &= -2 \int_0^\tau \exp\,[\gamma_1^2(\tau - \eta)]\, \frac{\partial T(y, \eta)}{\partial y}\, d\eta. \end{aligned} \tag{8}$$

The above statement may be examined by direct substitution of the functions (8) into Eqs. (6) and making use of Eq. (7). Bearing in mind the form of Eqs. (8) and assumptions (a)–(c) on the temperature field, one can establish the following inequality describing the properties of the functions $\mathfrak{F}_1$ and V_1 (see Appendix):

$$\begin{gathered} \mathfrak{F}_1 \leq 0, \qquad V_1 \geq 0, \\ \frac{\partial \mathfrak{F}_1}{\partial y} \geq 0, \qquad \frac{\partial \mathfrak{F}_1}{\partial \tau} \leq 0, \qquad \frac{\partial V_1}{\partial y} \leq 0, \qquad \frac{\partial V_1}{\partial \tau} \geq 0, \\ -2\frac{\partial T}{\partial y} \geq W(y, \tau, \gamma_1) \geq 0 \quad \text{and} \quad 0 \leq W(y, \tau, \gamma_1) \leq 2T(y, \tau), \\ -\mathfrak{F}_1(y, \tau, \gamma_1) \leq \gamma_1 V_1(y, \tau, \gamma_1) \quad \text{for} \quad \tau \geq 0, \quad y \geq 0, \quad 0 < \gamma_1 \leq 1, \end{gathered} \tag{9}$$

where

$$W(y, \tau, \gamma_1) = 2\gamma_1 T(y, \tau) - \gamma_1 \mathfrak{F}_1(y, \tau, \gamma_1) - \gamma_1^2 V_1(y, \tau, \gamma_1). \tag{10}$$

Also,

$$\begin{gathered} \lim_{y \to \infty} W[y, \Phi(y), \gamma_1] = 0, \\ -\lim_{y \to \infty} \mathfrak{F}_1[y, \Phi(y), \gamma_1] = \gamma_1 \lim_{y \to \infty} V_1[y, \Phi(y), \gamma_1], \\ -\lim_{y \to \infty} \mathfrak{F}_1(y, (y - \eta)/\gamma_1\,, \gamma_1) \leq 2 \exp\,(-\gamma_1 \eta), \end{gathered} \tag{11}$$

where $\Phi(y)$ is an arbitrary function (continuous and determined in $A \leq y < \infty$) which satisfies the condition $0 < \Phi(y) < y^m/4$, $0 \leq m < 2$, and η, A are constants.

The above properties are the basis for the analysis performed in the next section.

3. Analysis of the formation of various regions in the plane $\tau - y$ (Fig. 4). In order to obtain the solution for impact loading, let us first assume that $p_1(t)$ has the form shown by the dotted line in Fig. 2. The time t_E is regarded as a small quantity which will later be allowed to go to zero.

From Eq. (4) the following boundary condition for the function $\mathfrak{F}$ is obtained:

$$\mathfrak{F}|_{y=0} = -p(\tau) + 2T(0, \tau) = -n(\tau), \tag{12}$$

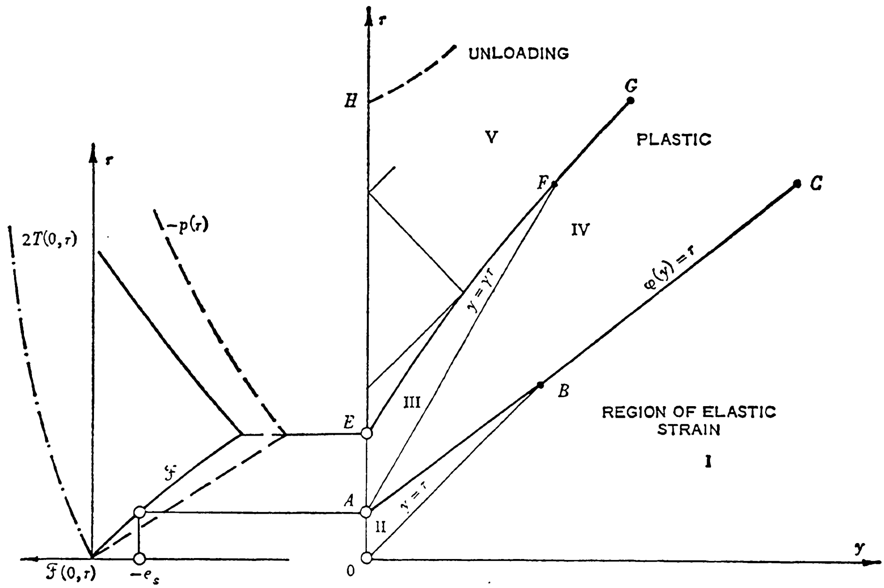

FIG. 4

where

$$p(\tau) = \frac{2p_1(k\tau/a_0^2)}{3K\alpha T_0}.$$

We shall consider the case in which the plastic deformation appears at the boundary $y = 0$, i.e.:

$$\sup\,[-2T(0, \tau) + p(\tau)] > e_s\,.$$

The left-hand side of the above inequality is assumed to considerably exceed the yield limit e_s (see Sec. 1). At the beginning of the process each point $y > 0$ is an elastic state. In the region of elastic strain (Fig. 4, region I) the solution takes the form

$$\mathfrak{F}(y, \tau) = \mathfrak{F}_1(y, \tau, 1), \qquad V(y, \tau) = V_1(y, \tau, 1). \tag{13}$$

From Eqs. (9) and from Eqs. (5) and (13), it follows that the function $\mathfrak{F}$ and S_1 are, in this region, monotonically decreasing, whereas the function V is monotonicallyincreasing. Moreover, since

$$\frac{d}{dy}\,\mathfrak{F}_1\!\left(y, \frac{y + \text{const}}{\gamma_1}, \gamma_1\right) = -W\!\left(y, \frac{y + \text{const}}{\gamma_1}, \gamma_1\right), \tag{14}$$

then the function $\mathfrak{F}$, which is the measure of the effective stress, decreases monotonically along the characteristic lines $y = \tau + \text{const}$.

If the density of heat flux transfered through the plane $y = 0$ is sufficiently large, then it is possible that elastic-plastic deformations appear to the right of the line $y = \tau$. From the properties of the function $\mathfrak{F}_1$ and from Eq. (14) it follows that this occurs if the following condition is fulfilled:

$$\lim_{y\to\infty} \mathfrak{F}_1(y, y, 1) < -e_s\,. \tag{15}$$

On the other hand, from Eq. $(11)_3$ it follows that this condition can be fulfilled only if

$$e_s < 2. \tag{16}$$

Thus for a given material, the plastic wave can occur to the right of the line $y = \tau$ only for sufficiently large T_0. Relation (16) is, of course, only a necessary condition; in addition the heat flux density in the initial stage of heating must be sufficiently high.

Let us now assume that this condition is fulfilled, and let us consider the function $\varphi(y) = \tau$ determined by the formula

$$\mathfrak{F}_1[y, \varphi(y), 1] = -e_s . \tag{17}$$

The following expression for $\varphi'(y)$ is obtained by calculation of the total derivative with respect to y of Eq. (17):

$$\varphi' = \frac{d\varphi}{dy} = 1 - \frac{W[y, \varphi(y), 1]}{e_s + 2T(y, \varphi)}. \tag{18}$$

Since $0 \leq W \leq 2T$, then from Eq. (18) it follows that

$$0 \leq \varphi' \leq 1.$$

Thus, Eq. (17) describes the plastic wave $\tau = \varphi(y)$ for $y \geq y_B$ (Fig. 4). Bearing Eq. $(11)_1$ in mind we get

$$\lim_{y\to\infty} \varphi'(y) = 1;$$

i.e., the line φ possesses the asymptote $y = \tau + \alpha_1$, where α_1 may be calculated from the expression

$$\lim_{y\to\infty} \mathfrak{F}(y, y - \alpha_1, 1) + e_s = 0.$$

One can show (see Eq. (A11)) that the parameter α_1 is bounded, i.e.,

$$\alpha_1 \leq \ln (2/e_s).$$

At the boundary $y = 0$ plastic deformation appears after a certain time τ_A, $n(\tau_A) = e_s$. Therefore the area bounded by lines $y = 0$, $y = \tau$ $(y < y_B)$ is the region of elastic strain $(A - B - 0$ in Fig. 4). The solution in this region is given by the formula

$$\mathfrak{F}(y, \tau) = \mathfrak{F}^{II}(y, \tau) = 2T(0, \tau - y) - p(\tau - y) - \mathfrak{F}_1(0, \tau - y, 1) + \mathfrak{F}_1(y, \tau, 1). \tag{19}$$

The function $\mathfrak{F}^{II}$ decreases along characteristic lines $y + \text{const} = \tau$. Moreover, if the derivative $\dot{T}(0, 0)$ exists, then we can always choose a slope ω_1 (Fig. 2) of the curve $p_1(t)$ such that $\dot{\mathfrak{F}}^{II}$ will be less than zero at each point in this region. This suggests that the equation for the plastic wave $\tau = \varphi(y)$ for $y < y_B$ can be obtained by substituting $\mathfrak{F}^{II} = -e_s$ into Eq. (19):

$$\mathfrak{F}^{II}[y, \varphi(y)] = -e_s . \tag{20}$$

In order to prove the above statement it is necessary and sufficient to satisfy the condition

$$-1 \leq \varphi'(y) \leq 1 \quad \text{for} \quad 0 \leq y < y_B . \tag{21}$$

By calculation of the total derivative of Eq. (20) we get

$$\varphi'(y) = 1 - \frac{W[y, \varphi(y), 1]}{n'(\varphi - y) + e_s - p(\varphi - y) + 2T(y, \varphi)}. \tag{22}$$

Now, taking into account that $W \geq 0$, Eq. (21) can be transformed into the equivalent form

$$n'(\varphi - y) \geq \tfrac{1}{2}W(y, \varphi, 1) - 2T(y, \varphi) - e_s + p(\varphi - y). \tag{23}$$

Hence, taking Eq. (9) into account, one can evaluate the slope ω_1 as follows:

$$\omega_1 k/(3K\alpha T_0 a_0^2) = \omega/2 \geq T(0, \tau) + \dot{T}(0, \tau). \tag{24}$$

Since the function $\dot{T}(0, \tau)$ is bounded then the above inequality can be satisfied throughout the time period $0 \leq \tau \leq \tau_E$ for sufficiently large ω. Thus, Eq. (20) describes the plastic wave in the segment $0 \leq y < y_B$.

Let us now assume that

$$\lim_{y \to \infty} \mathfrak{F}_1(y, y, 1) > -e_s . \tag{25}$$

The above condition is satisfied if $e_s > 2$. In this case the region $y \geq \tau$ is purely elastic. Since the function $\dot{T}(0, \tau)$ is bounded, then (24) is satisfied by a sufficiently large ω. The solution in region II (Fig. 5) is determined by Eq. (19) whereas the plastic wave $\varphi(y)$ is determined by Eq. (20). Taking into account Eqs. $(11)_1$ and (22), (23), it is easily seen that $\lim_{y\to\infty} \varphi'(y) = 1$; i.e., the plastic wave again possesses the asymptote $y = \tau - \alpha_2$. In this case the value of the coefficient α_2 can be calculated from the following expression:

$$p(\alpha_2) - 2T(0, \alpha_2) + \mathfrak{F}_1(0, \alpha_2, 1) = e_s + \lim_{y \to \infty} \mathfrak{F}_1(y, \alpha_2 + y, 1). \tag{26}$$

At time τ_E (Fig. 4) the unloading wave begins propagating from point E since the function $\dot{n}$ changes sign at this point. The shape of this wave can be determined by making use of numerical or approximate methods similar to that used for the isothermal process [1], [2].

The region AFE in Fig. 4 is the region of intensive elastic-plastic deformation. When (15) is satisfied ($e_s < 2$), secondary plastic strain appears at the boundary $y = 0$ after time τ_H (point H in Fig. 4); i.e., after time τ_H the wave of secondary plastic strain will propagate. Indeed, after the time $\tau = \tau^*$ we can assume $p(\tau) \approx 0$ and

$$\mathfrak{F}|_{y=0} \approx 2T(0, \tau) \quad \text{for} \quad \tau > \tau^*;$$

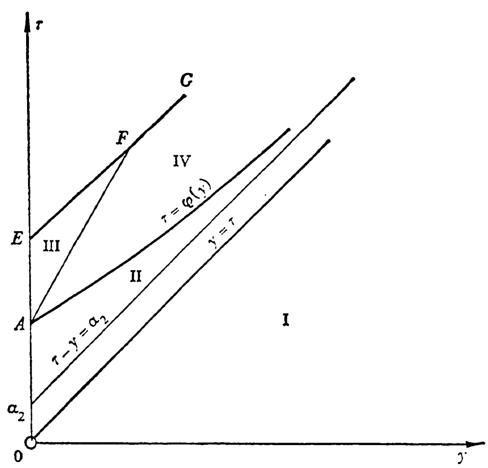

FIG. 5

but $\lim_{\tau \to \infty} T(0, \tau) = 1$; hence

$$\lim_{\tau \to \infty} \mathfrak{F} \,|_{y=0} = 2 > e_s \,.$$

The wave of secondary plastic strain can also, of course, appear if (15) is not satisfied.

It may also be noted that if the function $\dot{T}(0, \tau)$ is discontinuous at any point, then the function $\varphi'(y)$ is also discontinuous at the corresponding point (see Eq. (22)).

In order to establish completely the wave picture shown in Fig. 4, one should show that $\dot{\mathfrak{F}}$ is negative in the elastic-plastic regions III and IV. We shall prove this for the case of impact loading (see Sec. 4).

It may be observed that if $T(0, \tau)$ continuously approaches zero then (1) the point B moves along the line $y = \tau$ to infinity, (2) the line $\varphi(y)$ approaches the characteristic $y = \tau - \tau_A$ and (3) the unloading wave E-F becomes the unloading wave for the pure mechanical case ($T = 0$). Thus, the wave picture for a pure mechanical load can be obtained as a limit.

4. The limiting case $\tau_E \to 0$: impact loading. Taking into account the results of the previous analysis, it is easy to predict the wave picture for the case in which the function $p(\tau)$ is discontinuous at point $\tau = 0$. If point E approaches point 0, then the plastic waves A-B in Fig. 4 and $\varphi(y)$ in Fig. 5 approach the characteristic line $y = \tau$. These waves are transformed into a strong discontinuity wave $y = \tau$ with the value $\mathfrak{F} = -e_s$ at its front (Figs. 6 and 7). The stress jump disappears at point B (Fig. 6) if (15) is satisfied. The region AEF is transformed into the second strong discontinuity plastic wave $y = \gamma\tau$ (Figs. 6 and 7). This wave becomes simultaneosuly the unloading wave. At point F the stress jump disappears at the front of this wave.

We shall now show that the solution in the plane $\tau - y$ is unique. In the region GBC (Fig. 6) the Cauchy problem for the set of Eqs. 6 (taking $\gamma_1 = \gamma$) is determined. It

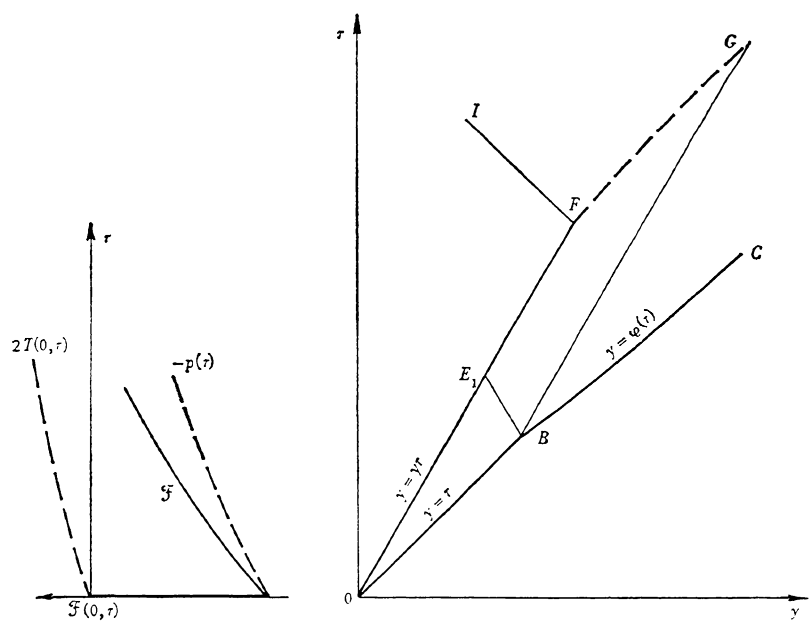

FIG. 6

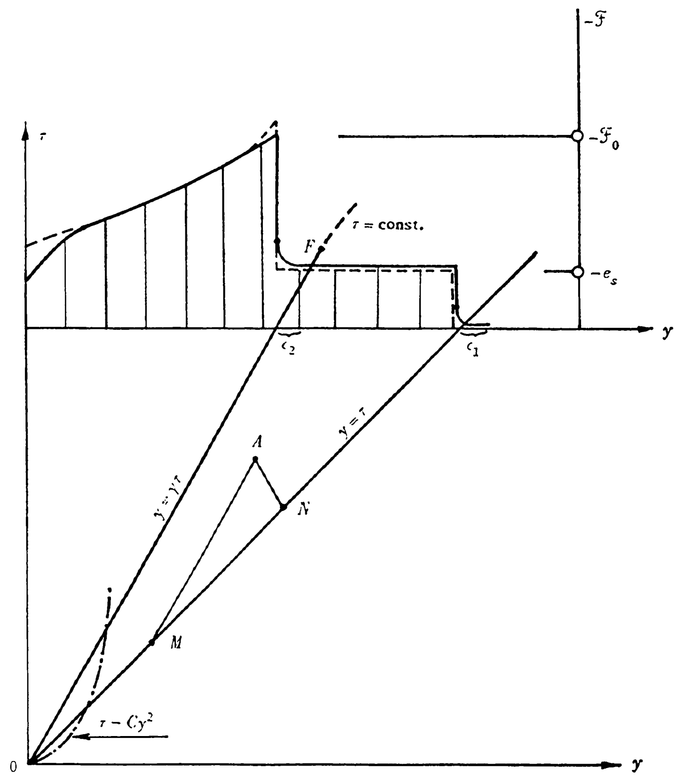

Fig. 7

follows from this that V and $\mathfrak{F}$ are prescribed along the noncharacteristic line B-C. At the front of the strong discontinuity wave $y = \tau$ the function $\mathfrak{F}$ is equal to $-e_s$ and the function V can be calculated from the dynamical and kinematical compatibility conditions. These are identical in the case considered. The line 0-B is noncharacteristic for the Eqs. 6 (when $\gamma_1 = \gamma$). Thus, the Cauchy problem is again determined in the region OBE_1. From the solution in the regions OBE_1 and GBC the values $\mathfrak{F}$ and V compatible with the relations along the characteristics on B-E_1 and B-G are known. The characteristic problem is determined in the region E_1BGF. From the dynamical and kinematical compatibility conditions which are again identical the relation between $\mathfrak{F}$ and V at the front of the wave O-E_1-F ($y = \gamma\tau - 0$) can be found. Hence, in the region τ-OE_1FI the mixed boundary problem for the set of Eqs. 6 (taking $\gamma_1 = 1$) is determined. The solution in the region IFG and the shape of the weak discontinuity unloading wave F-G must be determined simultaneously by using the condition of the continuity of $\mathfrak{F}$ and V along F-G. This leads to the well-known problem for the unloading wave [1].

Let us consider the function

$$
\begin{aligned}
2\mathfrak{F}(y, \tau) = 2\mathfrak{F}_1(y, \tau, \gamma) - 2e_s + \frac{1}{\gamma} W(y_N, y_N, \gamma) - 2(1-\gamma)T(y_N, y_N) \\
- \frac{1}{\gamma} W(y_M, y_M, \gamma) + 2(1-\gamma)T(y_M, y_M) \\
- 2\mathfrak{F}_1(y_M, y_M, \gamma) + \gamma W(y_M, y_M, 1) - \gamma W(y_N, y_N, 1),
\end{aligned}
\tag{28}
$$

which describes the solution in the region $\gamma\tau < y < \tau$ for the case when (25) is satisfied (Fig. 7). In Eq. (28) y_N and y_M are the functions of τ, y determined by the formulae

$$y_N = (y + \gamma\tau)/(1 + \gamma), \qquad y_M = (y - \gamma\tau)/(1 - \gamma).$$

After certain transformations (see Eq. (A2)) the time derivative of the above function $\mathfrak{F}$ can be expressed in the following form:

$$\frac{2}{\gamma}\frac{\partial \mathfrak{F}}{\partial \tau} = [W(y_N, y_N, \gamma) - W(y, \tau, \gamma)]$$
$$- [W(y, \tau, \gamma) + 2\gamma^2 V_1(y, \tau, \gamma) - W(y_M, y_M, \gamma) - 2\gamma^2 V_1(y_M, y_M, \gamma)] \tag{29}$$
$$- \frac{2\gamma}{1+\gamma} W(y_N, y_N, 1) - \frac{2\gamma}{1-\gamma} W(y_M, y_M, 1).$$

We shall now show that the above function is negative throughout the region considered.

It may easily be shown by using Eq. (A2) that the following relations hold:

$$\frac{d}{dy}\left[W\left(y, \frac{y-\eta}{\gamma}, \gamma\right) + 2\gamma^2 V_1\left(y, \frac{y-\eta}{\gamma}, \gamma\right)\right] = 2\dot{T}\left(y, \frac{y-\eta}{\gamma}\right),$$
$$\frac{d}{dy} W\left(y, \frac{\beta - y}{\gamma}, \gamma\right) = -2\dot{T}\left(y, \frac{\beta - y}{\gamma}\right), \tag{30}$$

where β, η are arbitrary constants. Taking the above into account, Eq. (29) can be rewritten in terms of the curvilinear integrals of $\dot{T}$ along the characteristic lines M-A and A-N (Fig. 7); i.e.:

$$-\frac{1}{\gamma}\frac{\partial \mathfrak{F}}{\partial \tau} = \oint_{MA} \dot{T}\, dy + \oint_{AN} \dot{T}\, dy + \frac{\gamma}{1+\gamma} W(y_N, y_N, 1) + \frac{\gamma}{1-\gamma} W(y_M, y_M, 1). \tag{31}$$

Since all members on the right-hand side of the above equation are greater then zero, $\dot{\mathfrak{F}}$ is negative in the region $\gamma\tau < y < \tau$. This proves that this region is one of active elastic-plastic strain. Eq. (28) can also be rewritten in terms of the curvilinear integrals:

$$\mathfrak{F} + e_s = -\frac{1}{\gamma}\oint_{MA} \dot{T}\, dy - \frac{1}{\gamma}\oint_{AN} \dot{T}\, dy + \frac{\gamma}{2}[W(y_M, y_M, 1) - W(y_N, y_N, 1)]$$
$$- (1-\gamma)T(y_N, y_N) - (1+\gamma)T(y_M, y_M) + 2T(y, \tau). \tag{32}$$

By use of the known solution for the isothermal case [1], the following expression determining $\mathfrak{F}$ at the front of the wave $y = \gamma\tau$ can be found:

$$\mathfrak{F}\,|_{\tau = y/\gamma + 0} = \mathfrak{F}_0(y) = -\frac{2\gamma}{1+\gamma}\sum_{i=0}^{\infty} r^i p\left(\frac{1-\gamma}{\gamma} r^i y\right)$$
$$+ \frac{2\gamma}{1+\gamma}\sum_{i=0}^{\infty} r^i \Psi\left(\frac{1-\gamma}{\gamma} r^i y\right) - \frac{2\gamma}{1-\gamma^2}\sum_{i+0}^{\infty} r^i f(r^i y) \tag{33}$$
$$+ \mathfrak{F}_1\left(y, \frac{y}{\gamma}, 1\right) + \frac{1}{1-\gamma} f(y),$$

where

$$r = \frac{1 - \gamma}{1 + \gamma} < 1, \qquad \Psi(z) = 2T(0, z) - \mathfrak{F}_1(0, z, 1)$$

$$f(z) = \frac{1}{\gamma} W\left(\frac{2z}{1+\gamma}, \frac{2z}{1+\gamma}, \gamma\right) - \frac{1}{\gamma} W\left(z, \frac{z}{\gamma}, \gamma\right) - \gamma W\left(\frac{2z}{1+\gamma}, \frac{2z}{1+\gamma}, 1\right)$$
$$+ \gamma W\left(z, \frac{z}{\gamma}, 1\right) - (1 - \gamma)\mathfrak{F}_1\left(z, \frac{z}{\gamma}, 1\right)$$
$$+ 2(1 - \gamma)T\left(z, \frac{z}{\gamma}\right) - 2(1 - \gamma)T\left(\frac{2z}{1+\gamma}, \frac{2z}{1+\gamma}\right).$$

By analysing Eqs. (31)–(33), as well as by taking into account the properties of the functions W and $\mathfrak{F}_1$, the shape of the wave at a given time can be predicted. In Fig. 7 the shape of the wave when $T = 0$ is shown by a dotted line. Beyond the narrow surface layer (curve $\tau = Cy^2$ in Fig. 7) the waves $y = \tau$ and $y = \gamma\tau$ propagate in an essentially cold medium. For the region of large temperature gradients $0 \le y \le C^{-1/2}\tau^{1/2}$ the addition of the effects of dynamical thermal expansion to those of dynamical mechanical load leads to a decrease in the jump of the function $\mathfrak{F}$ at the front of the strong discontinuity waves $y = \tau$ and $y = \gamma\tau$. In the region $0 \le y < \gamma\tau$ the function $\mathfrak{F}$ decreases from $-p(\tau) + 2T(0, \tau)$ to some value greater (for small τ) than the value obtained in the case $T = 0$ (as may be seen by calculation of $d\,\mathfrak{F}_0/dy$ at $y = 0$). At the point $y = \gamma\tau$ it suddenly increases to a value less than $-e_s$. In the narrow region $\gamma\tau < y \le \epsilon_2$ (ϵ_2 is of the order $C^{-1/2}$) the solution continuously increases to a value close to $-e_s$, whereas in the region $\gamma\tau + \epsilon_2 \le y < \tau$ it is almost equal to $-e_s$. At the point $y = \tau$ it again increases suddenly to some value greater than zero. In the region $\tau < y < \tau + \epsilon_1$ (ϵ_1 is also of the order $C^{-1/2}$) it changes continuously to a value close to zero. In the region $y > \tau + \epsilon_1$ the solution is almost equal to zero.

The difference between the solution above and that for the case $T = 0$ at the boundary $y = 0$ (Fig. 7) may also be obtained from an approximate analysis by assuming that the wave caused by the mechanical impact propagates in the quasistatic field of the thermal stresses.

Remark. The case for which the function $\dot{T}|_{y=0}$ is infinite at point $\tau = 0$ should be considered separately. The approximate analysis performed by the author suggests that the final wave picture shown in Fig. 7 is also valid for this case.

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Appendix. The proofs of properties (9) and (11) and some other statements about $\mathfrak{F}_1$ and V_1 are given here.

We must first recall the following properties of the solution of the heat conduction equation which arise directly from the principle of upper and lower bounds on the temperature θ.

If the bounded function $\theta(y, \tau)$ in the region $0 < y < \infty$, $0 < \tau < \infty$ satisfies the homogeneous heat conduction equation and reaches the prescribed initial values $\theta|_{\tau=0} = 0$ and boundary values $\theta|_{y=0} = q(\tau)$ continuously, where $q(\tau)$ is a positive and nondecreasing function, then

$$\begin{aligned} &\partial\theta/\partial y \le 0, \qquad \partial\theta/\partial\tau \ge 0 \\ &\qquad 0 \le \theta \le [\sup q(\tau)] \operatorname{erfc}(y/2\tau^{1/2}) \quad \text{for} \quad y \ge 0, \quad \tau \ge 0, \end{aligned} \tag{A1}$$

where

$$\operatorname{erfc}(z) = 2\pi^{-1/2} \int_z^\infty \exp(-t^2)\, dt.$$

Since in our case $T(0, \tau) \le 1$, the inequality $(9)_1$ arises directly from Eqs. (8), taking into account the assumption about $T(0, \tau)$ and Eq. (A1).

By calculation of the derivatives of V_1 and $\mathfrak{F}_1$ with respect to y and τ we get

$$\begin{aligned} \frac{\partial \mathfrak{F}_1}{\partial \tau} &= \gamma_1^2 \mathfrak{F}_1 - 2\gamma_1^2 T, \qquad & \frac{\partial V_1}{\partial \tau} &= \gamma_1^2 V_1 - 2\frac{\partial T}{\partial y} \\ \frac{\partial \mathfrak{F}_1}{\partial y} &= \gamma_1^2 V_1\,, & \frac{\partial V_1}{\partial y} &= \mathfrak{F}_1 - 2T. \end{aligned} \tag{A2}$$

The inequalities $(9)_2$ arise from Eqs. (A2) and $(9)_1$. The function

$$W(y, \tau, \gamma_1) = 2\gamma_1 T(y, \tau) - \gamma_1 \mathfrak{F}_1(y, \tau, \gamma_1) - \gamma_1^2 V_1(y, \tau, \gamma_1) \tag{A3}$$

reaches the following boundary values:

$$W(y, 0, \gamma_1) = 0, \qquad \lim_{y\to\infty} W = 0$$

and satisfies the following equation:

$$\frac{\partial W}{\partial y} = \gamma_1 W + 2\gamma_1 \frac{\partial T}{\partial y}. \tag{A4}$$

The above statement may be checked by using Eqs. (A2). Thus, W may be expressed in the form

$$\begin{aligned} W &= -2\gamma_1 \int_y^\infty \exp[\gamma_1(y - x)] \frac{\partial T(x, \tau)}{\partial x}\, dx \\ &= 2\gamma_1 T(y, \tau) - 2\gamma_1^2 \int_y^\infty \exp[\gamma_1(y - x)] T(x, \tau)\, dx. \end{aligned} \tag{A5}$$

Bearing in mind that $T(y, \tau)$ satisfies the homogeneous heat conduction equation, we can transform Eq. (A5) into:

$$W + 2\frac{\partial T}{\partial y} = -2\int_y^\infty \exp[\gamma_1(y - x)]\dot{T}(x, \tau)\, dx. \tag{A6}$$

Since $\dot{T} \ge 0$ and $(\partial T/\partial y) \le 0$, then from Eqs. (A5), (A6) we get

$$0 \le W \le -2\frac{\partial T}{\partial y} \quad \text{and} \quad 0 \le W \le 2\gamma_1 T(y, \tau). \tag{A7}$$

Taking into account the form of the function W from $(A7)_2$, we also get

$$-\mathfrak{F}_1(y, \tau, \gamma_1) \le \gamma_1 V_1(y, \tau, \gamma_1). \tag{A8}$$

This is the proof of $(9)_3$ and $(9)_4$.

Since the following equality holds:

$$\lim_{y\to\infty} \operatorname{erfc}\left[\frac{y}{2\sqrt{\Phi(y)}}\right] = 0,$$

where $\Phi(y)$ is the continuous and defined function in $A \leq y < \infty$ which satisfies the condition

$$0 < \Phi(y) < y^m/4, \qquad 0 \leq m < 2,$$

then from Eqs. (A7) and (A1) it follows that

$$\begin{aligned} \lim_{y\to\infty} W[y, \Phi(y), \gamma_1] &= \lim_{y\to\infty} T[y, \Phi(y)] = 0, \\ -\lim_{y\to\infty} \mathfrak{F}_1[y, \Phi(y), \gamma_1] &= \gamma_1 \lim_{y\to\infty} V_1[y, \Phi(y), \gamma_1]. \end{aligned} \tag{A9}$$

The above proves Eqs. $(11)_1$ and $(11)_2$.

From (A1) it follows that

$$\begin{aligned} -\mathfrak{F}_1(y, \tau, \gamma_1) \leq 2\gamma_1^2 \int_0^\tau \exp[\gamma_1^2(\tau - t)] \operatorname{erfc}(y/2t^{1/2})\, dt &= -2 \operatorname{erfc}(y/2\tau^{1/2}) \\ &+ \exp[\gamma_1(\gamma_1\tau - y)] \operatorname{erfc}\left[\frac{y - 2\gamma_1\tau}{2\tau^{1/2}}\right] \\ &+ \exp[\gamma_1(y + \gamma_1\tau)] \operatorname{erfc}\left[\frac{y + 2\gamma_1\tau}{2\tau^{1/2}}\right] = L(y, \tau, \gamma_1) \end{aligned}$$

but

$$\lim_{y\to\infty} L\left(y, \frac{y - \eta}{\gamma_1}, \gamma_1\right) = 2 \exp(-\gamma_1\eta).$$

Hence, we get the proof of Eq. $(11)_3$,

$$-\lim_{y\to\infty} \mathfrak{F}_1\left[y, \frac{y - \eta}{\gamma_1}, \gamma_1\right] \leq 2 \exp(-\gamma_1\eta). \tag{A10}$$

If α is the root of

$$-\lim_{y\to\infty} \mathfrak{F}_1\left(y, \frac{y - \alpha}{\gamma_1}, \gamma_1\right) = N,$$

then from Eq. (A10) we get

$$\alpha \leq \frac{1}{\gamma_1} \ln(2/N). \tag{A11}$$

It is worth emphasizing that $\mathfrak{F}_1$, V_1 and W also satisfy the homogeneous heat conduction equation. Moreover, they can be expressed in terms of the exponential function and a sutiably defined mean temperature or mean density of the heat flux.

Since all mathematical operations carried out in the Appendix apply to any point in the interior of the plane $\tau > 0$, $y > 0$ where the function T is analytical, then all the above remarks are also valid for the case of discontinuous $T(0, \tau)$.

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