NECESSARY AND SUFFICIENT CONDITIONS FOR BOUNDED GLOBAL STABILITY OF CERTAIN NONLINEAR SYSTEMS*

BY

R. K. BRAYTON

IBM Thomas J. Watson Research Center
Yorktown Heights, New York

Abstract. Nonlinear systems having the form

$$\dot{x} = -Ax + By$$
$$\dot{y} = Cx - f(y),$$

where $\partial f/\partial y$ is a symmetric matrix, are considered. Such systems include the class of nonlinear reciprocal networks where the nonlinearity is voltage (or current) controlled. Also included, provided $c^T b \neq 0$, are the equations of nonlinear feedback systems,

$$\dot{x} = Ax + bf(c^T x),$$

considered by Aizerman [1]. A type of stability called bounded global stability is considered which requires that all bounded solutions decay as $t \to \infty$ to the set of equilibrium points. A necessary and sufficient condition on the linear parts of these systems for their bounded global stability is given. It is also shown that this condition insures the existence of at least one stable equilibrium point.

1. Introduction. In 1947, Aizerman [1] stated the following conjecture:

Aizerman's Conjecture. The system

$$\dot{x} = Ax + b\tilde{f}(c^T x) \tag{1}$$

is absolutely stable for all $\tilde{f}(y)$, $k_1 \leq \tilde{f}(y)/y \leq k_2$, if and only if the system

$$\dot{x} = Ax + hbc^T x \tag{2}$$

is absolutely stable for all h, $k_1 \leq h \leq k_2$.

This conjecture stood for eleven years until Pliss [2] in 1958 constructed a counterexample. A simpler counterexample was constructed later by Willems [3]. Thus the attempt completely to characterize the behavior of the nonlinear system (1) in terms of its linearized counterpart (2) failed.

In this paper we consider a class of nonlinear systems for which a result similar to Aizerman's Conjecture holds. These systems were considered by Moser [4] and have the form

$$\begin{aligned} \dot{x} &= -Ax + By \\ \dot{y} &= Cx - f(y) \end{aligned} \tag{3}$$

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where $\partial f/\partial y$ is symmetric. This class includes nonlinear reciprocal electrical networks where the nonlinearity is voltage controlled (or current controlled). Such networks have differential equations with the form

$$\begin{aligned} L\, di/dt &= -Ai + \alpha v \\ C\, dv/dt &= -\alpha^T i - f(v) \end{aligned} \tag{4}$$

where A, L, C and $\partial f/\partial v$ are symmetric (see [5]). The transformation[1]

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} L^{1/2} & 0 \\ 0 & C^{1/2} \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix}$$

maps these equations into the form (3).

The differences in the Eqs. (3) from the feedback equations (1) considered by Aizerman are that in (3),

(a) f is not necessarily scalar.

(b) No restriction of the form $k_1 \leq f(y)/y \leq k_2$ is imposed.

(c) The type of stability that is considered is that all bounded solutions decay to the set of equilibrium points.[2] Thus more than one equilibrium point is allowed as well as unbounded solutions. The absolute stability considered by Aizerman required that all solutions decay to the single equilibrium point $x = 0$.

(d) A symmetry condition, $C(A + i\omega I)^{-1}B$ symmetric for all ω, is imposed. If f is scalar this is no restriction.

(e) If f is scalar and $b \not\equiv 0$ in (1), then (1) can be transformed into (3) by a nonsingular linear transformation if and only if $c^T b \neq 0$. Then $f(y) = \beta y + c^T b \tilde{f}(y)$ for some constant β.

The main result of this paper is

MAIN THEOREM. *Given A, B, C, where $e^{-At} \in L_2[0, \infty)$, and $C(A + i\omega I)^{-1}B$ symmetric for all ω, then the following statements are equivalent.*

(i) *The system* (3) *is bounded globally stable for all $f(y)$, $\partial f/\partial y$ symmetric.*

(ii) *The linearization of* (3) *with $f(y) = Dy$ is bounded globally stable for all D, D symmetric.*

(iii) *For all $\omega \neq 0$ and all ζ, $\zeta \neq 0$,*

$$\bar{\zeta}^T(I + C(A^2 + \omega^2 I)^{-1}B)\zeta > 0.$$

Thus if $c^T b \neq 0$ and no restriction is made on $\tilde{f}$, then Aizerman's Conjecture is valid for the type of stability considered in this paper.

The motivation for guessing that (i) and (ii) are equivalent, even in view of the fact that Aizerman's Conjecture is false, is that in [5] the following two results were proved.

RESULT 1. [5, Theorem 3, p. 19.] If $\|A^{-1}B\| < 1$, $B = -C^T$, and A positive definite, then (3) is bounded globally stable.

RESULT 2. [5, Theorem 6, p. 30.] If $B = -C^T$, A symmetric, and $f(y) = Dy$, D symmetric, then any nonreal eigenvalue λ of

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[1] L and C are positive definite.

[2] This type of stability will be called *bounded global stability* in this paper.

$$\begin{bmatrix} -A & B \\ -B^T & -D \end{bmatrix}$$

with eigenvector $\binom{x}{y}$ must lie on the circle

$$|\lambda|^2 \, |A^{-1}x|^2 + 2 \operatorname{Re} (\lambda) x^T A^{-1} x = |A^{-1}By|^2 - |x|^2.$$

Further if $||A^{-1}B|| < 1$, then this circle must lie in the plane $\operatorname{Re} (\lambda) < 0$.

Thus the condition $||A^{-1}B|| < 1$ which guarantees bounded global stability for the nonlinear system also guarantees that the eigenvalues of the linearized system can never be purely imaginary. That this is the same condition suggests that our main theorem might be valid.

The proof of the main theorem is given in Sec. II. In Sec. III, this result is applied to nonlinear reciprocal networks and it is shown that (i), (ii), and (iii) are also equivalent to

(iv) *the linear part of the network, as viewed by the nonlinear part, appears capacitive at all frequencies.*
In [4], the condition (iii) was given by Moser and shown to be sufficient for bounded global stability.

In Sec. IV we show that this same condition implies that there must always exist at least one stable equilibrium point of the nonlinear system (3). By constructing an example where the capacitive condition (iii) fails and where the nonlinear system has three equilibrium points which are all unstable, this condition is shown to be necessary.

II. Proof of main theorem. It is clear that (i) implies (ii) since the case $f(y) = Dy$ is included in (i).

To demonstrate that (ii) implies (iii), we assume that (iii) does not hold. Then there exist $\zeta \neq 0$ and $\omega \neq 0$ such that

$$\bar{\zeta}^T (I + C(A^2 + \omega^2 I)^{-1} B) \zeta \leq 0.$$

Since as $\omega^2 \to \infty$, $I + C(A^2 + \omega^2 I)^{-1} B$ becomes positive definite, there must exist $\hat{\omega}^2 \geq \omega^2$ such that

$$\det (I + C(A^2 + \hat{\omega}^2 I)^{-1} B) = 0.$$

Here we have used that $C(A^2 + \hat{\omega}^2 I)^{-1} B$ is symmetric since

$$\operatorname{Im} C(A + i\omega I)^{-1} B = -\omega C(A^2 + \omega^2 I)^{-1} B.$$

Let φ be the zero real eigenvector, i.e.

$$(I + C(A^2 + \hat{\omega}^2 I)^{-1} B)\varphi = 0.$$

Let $D = C(A + \hat{\omega}^2 A^{-1})^{-1} B$ which is symmetric since

$$\operatorname{Re} C(A + i\omega I)^{-1} B = C(A + \omega^2 A^{-1})^{-1} B.$$

Then the system

$$\dot{x} = -Ax + By, \qquad \dot{y} = Cx - Dy,$$

which is the linearization of (3) with $f(y) = Dy$, has a nonconstant periodic solution. This follows since it is easily computed that

$$\begin{bmatrix} -A & B \\ C & -D \end{bmatrix} \begin{bmatrix} (A + i\hat{\omega}I)^{-1}B\varphi \\ \varphi \end{bmatrix} = i\hat{\omega} \begin{bmatrix} (A + i\hat{\omega}I)^{-1}B\varphi \\ \varphi \end{bmatrix}$$

i.e. the matrix $\binom{-A\ B}{C\ -D}$ has a purely imaginary eigenvalue. Thus a linearization exists (with D symmetric) which is not bounded globally stable.

To prove that (iii) implies (i), we use the following result of Moser [4].

Lemma (Moser). *Assume that $e^{-At} \in L_2[0, \infty)$ and $CA^{-1}B$ and $\partial f/\partial y$ are symmetric. If there exists a $\delta > 0$ such that*

$$\bar{\zeta}^T(I + C(A^2 + \omega^2 I)^{-1}B)\zeta \geq \delta\bar{\zeta}^T\zeta \tag{5}$$

for all $\zeta \neq 0$ and all ω, then any bounded solution $y(t)$ of (3) has $\dot{y}(t) \in L_2[0, \infty)$, and $y(t)$ approaches the bounded part of the limit set of the system $\dot{z} = f(z) - CA^{-1}Bz$.

In fact, in the result given by Moser it is assumed that the function $G(y) = \int_0^y f(y)\, dy - \frac{1}{2}y^TCA^{-1}By$ has the property that $G(y) \to \infty$ as $|y| \to \infty$. However, it is easy to see that this is only used to obtain boundedness and is not necessary if the boundedness of y can be obtained by some other means or is assumed as in the lemma as stated above.

Now assume that (iii) holds. This is equivalent to (5), since $\bar{\zeta}^T\zeta + \bar{\zeta}^TC(A^2 + \omega^2I)^{-1}B\zeta$ is continuous in ω^2 and approaches $\bar{\zeta}^T\zeta$ as $\omega^2 \to \infty$. Since $C(A + i\omega I)^{-1}B$ is symmetric for all ω, then $CA^{-1}B$ is symmetric. Using Moser's lemma, any bounded solution $y(t)$ approaches the bounded part of the limit set of the system $\dot{z} = f(z) - CA^{-1}Bz$. The function

$$L(z) = -\int_0^z f(z)\, dz + \tfrac{1}{2}z^TCA^{-1}Bz,$$

satisfies $dL/dt = -\dot{z}^T\dot{z} \leq 0$; hence, $L(z)$ is a Liapunov function in the sense of LaSalle [6]. The set z where $dL/dt = 0$ is the set where $\dot{z} = 0$; i.e., $f(z) = CA^{-1}Bz$. Hence, by LaSalle's theorem [6], $y(t)$ approaches the set of points y where $f(y) = CA^{-1}By$. From (3) we have

$$x(t) = e^{-At}x(0) + A^{-1}By(t) - A^{-1}e^{-At}By(0) - \int_0^t e^{-A(t-\tau)}A^{-1}B\dot{y}(\tau)\, d\tau.$$

Since $\dot{y} \in L_2[0, \infty)$ and $e^{-At} \in L_2[0, \infty)$ the integral must approach zero with $t \to \infty$ and, hence,

$$\lim_{t\to\infty} x(t) = A^{-1}B \lim_{t\to\infty} y(t).$$

Thus $\dot{x} \to 0$ as $t \to \infty$, and

$$\lim_{t\to\infty} \dot{y} = \lim_{t\to\infty} [Cx - f(y)] = \lim_{t\to\infty} [CA^{-1}By - f(y)] = 0.$$

Hence, any bounded solution of (3) approaches its equilibrium set. Q.E.D.

Note that condition (ii) does not rule out eigenvalues of $\binom{-A\ B}{C\ -D}$ in the right half plane. In fact, since D is arbitrary, it is always possible to find a $-D$ large enough which makes the above matrix unstable. If we call this matrix $M(D)$, the condition (iii) is the requirement that the only way the eigenvalues of $M(D)$ can cross from the left-halfplane to the right-halfplane is through the origin (see Fig. 1).

III. Reciprocal networks and the capacitive condition. Now we shall see how reciprocal networks fall into the domain of our main theorem. Let L, C_1, C_2 be positive definite symmetric constant matrices. We assume that the nonlinearities are controlled

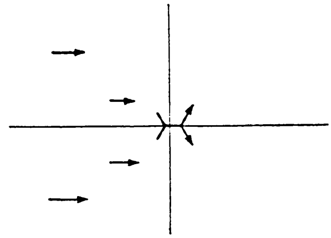

FIG. 1. Eigenvalues as functions of D.

by the voltages v_2 and enter only in the v_2 equations. Thus, the equations of motion are

$$\begin{aligned} L\,di/dt &= -A_1 i + \gamma_1 v_1 + \gamma_2 v_2\,, \\ C_1\,dv_1/dt &= -\gamma_1^T i - B_1 v_1 - \gamma_3 v_2\,, \\ C_2\,dv_2/dt &= -\gamma_2^T i - \gamma_3^T v_1 - g(v_2), \end{aligned} \tag{6}$$

where L, C_1, C_2, A_1, B_1 and $\partial g/\partial v_2$ are symmetric. We have neglected constant terms which can be removed by translating i, v_1, and v_2. Using the transformation

$$x = \begin{bmatrix} L^{1/2} & i \\ C_1^{1/2} & v_1 \end{bmatrix}, \qquad y = C_2^{1/2} v_2\,,$$

we have the system (3), where $A = -JM_1$, $B = JM_2$, $C = M_2^T$ and

$$J = \begin{bmatrix} -I_r & 0 \\ 0 & I_s \end{bmatrix}, \qquad M_1 = \begin{bmatrix} L^{-1/2}A_1L^{-1/2} & -L^{-1/2}\gamma_1 C_1^{-1/2} \\ -C_1^{-1/2}\gamma_1^T L^{-1/2} & C_1^{-1/2}B_1C_1^{-1/2} \end{bmatrix},$$

$$f(y) = C_2^{-1/2} g(C_2^{-1/2} y), \qquad M_2 = \begin{bmatrix} L^{-1/2}\gamma_2 C_2^{-1/2} \\ C_1^{-1/2}\gamma_3 C_2^{-1/2} \end{bmatrix}.$$

Here, r and s are the dimensions of i and v_1. Note that M_1 and $\partial f/\partial y$ are symmetric. To apply our theorem, we have to check that $C(A + i\omega I)^{-1}B$ is symmetric for all ω. But this is

$$M_2^T(-M_1 + i\omega J)^{-1} M_2\,,$$

which is symmetric since M_1 and J are symmetric. Then if $e^{JM_1 t} \in L_2[0, \infty)$,[3] we have bounded global stability if and only if

$$\bar{\zeta}^T\zeta + \bar{\zeta}^T M_2^T(M_1JM_1 + \omega^2 J)^{-1} M_2 \zeta > 0 \tag{7}$$

for all $\zeta \neq 0$ and ω.

This condition has a network interpretation in terms of reactive power. In (6) replace $g(v_2)$ by $-I(t)$ where $I(t)$ is periodic with period ω. If (6) *then* has a periodic solution with period ω, we can take the Fourier transform of (6) and solve for $F(I)$ in terms of $F(V_2)$($F(\ )$ denotes the Fourier transform). Computing the reactive power, we have

$$\begin{aligned} \text{Reactive Power} &\equiv (1/\omega)\ \text{Im}\ \bar{F}(v_2)\dot{\bar{F}}(I) \\ &= -\bar{F}(v_2)C_2^{1/2}[I + M_2^T(M_1JM_1 + \omega^2 J)^{-1}M_2]C_2^{1/2}F(v_2). \end{aligned}$$

[3] This is really no restriction for physical electrical networks with positive resistors since it is equivalent to the system (6) being stable for v_2 a constant.

Hence, the condition (7) is simply that the reactive power is negative for all ω. In other words, if we look into the linear part of the network through the ports that are connected to the nonlinearities, *the network appears capacitive.*

This result agrees with a well-known situation. Suppose we have a reciprocal network with no inductors, only capacitors and resistors. Then the equations are [5]

$$C(v)\, dv/dt = \partial P/\partial v. \tag{8}$$

Clearly, $P(v)$ is a Liapunov function and, hence, we have stability in the sense of LaSalle [6]. Our theorem says simply that if the network appears capacitive to the nonlinear resistors, then the situation is analogous to (8), and *there exists no other situation in which we can have a stability condition independent of the nonlinearity.*

Finally, we specialize Theorem 2 to the case described in [5, Theorem 3, p. 19], and restrict $L(i)$, $C(v)$ to be *constant* positive symmetric matrices. Then $J = -I$, and (7) becomes

$$\bar{\zeta}^T\zeta - \bar{\zeta}^T M_2(M_1^2 + \omega^2 I)^{-1} M_2 \zeta > 0$$

where M_1 is symmetric and positive definite. Since $\bar{\zeta}^T M_2(M_1^2 + \omega^2 I)^{-1} M_2\zeta$ takes its maximum at $\omega = 0$, the requirement is that

$$\|M_1^{-1} M_2\| < 1.$$

Since $M_1 = L^{-1/2} A_1 L^{-1/2}$, $M_2 = -L^{-1/2}\gamma_2 C_2^{-1/2}$ this condition is

$$\|L^{1/2} A_1^{-1}\gamma_2 C^{-1/2}\| < 1,$$

which is the condition given in [5, Theorem 3, p. 19]. *Therefore, this condition is necessary and sufficient for bounded global stability when L and C are constant.*

The condition $C(A + i\omega I)^{-1}B$ symmetric can be interpreted as the requirement that the linear part of the network at the nonlinear ports appears reciprocal. Thus the network can be allowed to have nonreciprocal components as long as they do not appear nonreciprocal to the nonlinear ports. Note that if there is only one nonlinear port, then this condition is automatically satisfied since then $C(A + i\omega I)^{-1}B$ is a scalar. *Hence, any network* (*reciprocal or not*) *with only one nonlinearity is bounded globally stable if and only if all linearizations are bounded globally stable.*

IV. Conditions which guarantee at least one stable equilibrium point.

Theorem 2. *There always exists at least one stable equilibrium point for the equations* (3) *under the assumptions that*

(i) $\partial f/\partial y$, *and* $CA^{-1}B$ *are symmetric,*
(ii) $\int_0^y f(y)\, dy - \frac{1}{2} y^T CA^{-1}By \to \infty$ as $|y| \to \infty$,
(iii) $\bar{\zeta}^T\zeta + \bar{\zeta}^T C(A^2 + \omega^2 I)^{-1} B\zeta > 0$ *for* $\zeta \neq 0$, $\omega \neq 0$, *and*
(iv) $e^{-At} \in L_2[0, \infty)$.

Remark. This theorem would seem to be necessary, since under these assumptions Moser [4] showed that all solutions of (3) were bounded and decayed to the equilibrium set. This is probably not possible if there are no stable equilibrium points.

Proof. Let $G(y) \equiv \int_0^y f(y)\, dy - \frac{1}{2} y^T CA^{-1}By$. Then the equilibrium solutions of (3) necessarily satisfy the equation $\partial G/\partial y = f(y) - CA^{-1}By = 0$. Since $G(y) \to \infty$ as $|y| \to \infty$, $G(y)$ must have a minimum. At this minimum, say $y = y_0$, we have that

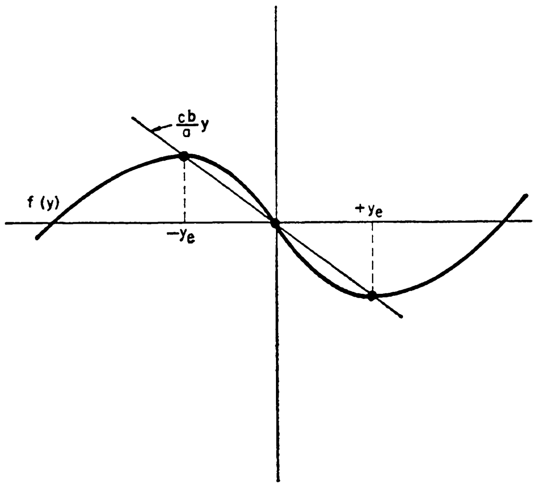

FIG. 2. Equilibrium points.

$(\partial f/\partial y)(y_0) - CA^{-1}B$ is positive semidefinite. Therefore, all of its eigenvalues are ≥ 0. Let $D_0 = (\partial f/\partial y)(y_0)$. Now we will show that the matrix

$$M(D) \equiv \begin{bmatrix} -A & B \\ C & -D \end{bmatrix}$$

for $D = D_0$ has all its eigenvalues ≤ 0, which will prove the theorem.

From assumption (iii) and the first theorem, we know that $M(D)$ has no purely imaginary eigenvalues. The only way eigenvalues of $M(D)$ can cross over from the left-halfplane to the right-halfplane as D varies is that the eigenvalue pass through the origin. But the matrix $N(D) \equiv D - CA^{-1}B$ has a zero eigenvalue if and only if $M(D)$ has a zero eigenvalue since

$$\det M(D) = \det(-A) \det(-D + CA^{-1}B)$$

and A is nonsingular.

Further, for D very large, $M(D)$ has all its eigenvalues in the left-halfplane. We let $D = D_0 + \alpha I$. The eigenvalues of $N(D)$ as functions of α can only pass from left to right through the origin as α increases. This can be seen by the following argument. Let $(D_0 + \alpha I - CA^{-1}B)y = \lambda y$ where λ and y are real since $(D_0 + \alpha I - CA^{-1}B)$ is symmetric. The derivative with respect to α yields

$$(D_0 + \alpha I - CA^{-1}B)y_\alpha + y = \lambda_\alpha y + \lambda y_\alpha .$$

Evaluation at any α_0 such that $\lambda(\alpha_0) = 0$ yields

$$(D_0 + \alpha_0 I - CA^{-1}B)y_{0\alpha} + y_0 = \lambda_{0\alpha} y_0$$

Since $y_0^T(D_0 + \alpha_0 I - CA^{-1}B) = 0$ we have $y_0^T y_0 = \lambda_{0\alpha} y_0^T y_0$ i.e. $\lambda_{0\alpha} = 1 > 0$. Hence, since $N(D_0) \geq 0$, then $N(D_0 + \alpha I) > 0$ for $\alpha > 0$.

Now suppose that $M(D_0)$ has an eigenvalue λ with $\mathrm{Re}\,\lambda > 0$. Since $M(D_0 + \alpha I) < 0$ for α large enough and $M(D_0 + \alpha I)$ can have no purely imaginary eigenvalue, then there must exist $\hat{\alpha} > 0$ such that $\det M(D_0 + \hat{\alpha}I) = 0$. But then $\det N(D_0 + \hat{\alpha}I) = 0$ which is a contradiction. Q.E.D.

It is interesting that if assumption (iii) is not imposed, then the conclusion of Theorem 2 can be false. The following example illustrates this:

$$\dot{x} = -ax + by, \qquad \dot{y} = cx - f(y),$$

where x and y are scalars. The equilibrium is given by $(cb/a)y = f(y)$. Choose $a > 0$ and $cb < 0$ so that $cb + a^2 < 0$. Let $f(y)$ be such that $f'(0) \equiv d_0$ and $f'(\pm y_e) \equiv d_1$ ($\pm y_e$ are the other equilibrium points as shown in Fig. 2) where $d_0 < cb/a < d_1 < -a$. One can easily compute that the matrix (associated with the linearization at $\pm y_e$) $\left(\begin{smallmatrix} -a & b \\ c & -d_1 \end{smallmatrix}\right)$ has two positive eigenvalues and the matrix (associated with $y = 0$) $\left(\begin{smallmatrix} -a & b \\ c & -d_0 \end{smallmatrix}\right)$ has one positive eigenvalue. Hence, all three equilibrium points are *unstable*. However, if the condition $1 + c(a^2 + \omega^2)^{-1}b > 0$ of Theorem 2 were imposed, i.e. $a^2 + cb > 0$, this situation could not happen.

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