HIGHER APPROXIMATIONS FOR TRANSONIC FLOWS*

BY

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Abstract. Using the results of the WKBI method, two hypothetical gases are introduced, whose graphs in the pressure-density plane and that of the polytropic gas have contact of order 4 and 5 at the sonic point. This is in contrast to the fact that such graphs for the Tricomi gas and the generalized Tricomi gas have contact of order 2 and 3, respectively, to that of the polytropic gas there. Various relations for these gases are derived and compared to those of the air, the Tricomi gas and the generalized Tricomi gas. Applicable range of the approximations to the airflow are $0.65 < M < 1.4$ for the first approximation, and $0.5 < M < 1.5$ for the second approximation, M being the local Mach number. This is compared to such ranges as $0.9 < M < 1.2$ for the Tricomi gas, and $0.75 < M < 1.3$ for the generalized Tricomi gas. Flow solutions for the hypothetical gases are expressed by the Airy functions.

For practical applications, emphasis is placed in the subsonic range. For such purposes, it is proved that changing the values of integration constants results in the applicable range of $0.45 < M < 1.35$ and $0.35 < M < 1.4$ for the first and the second gas, respectively.

1. Introduction. To analyze transonic flows by the hodograph method, the Tricomi approximation or the generalized Tricomi approximation is widely used to avoid mathematical complication of handling the exact solutions expressed by the hypergeometric functions. By doing so, the transonic similarity rule is derived, and the solutions can be expressed by the Airy functions. For details of these approximations, readers are referred to Ferrari and Tricomi [5]. Unfortunately, the applicable range of such approximations is rather limited. In this paper, attempts are made to extend the applicable range substantially without introducing such functions as hypergeometric or confluent hypergeometric functions.

2. Fundamental equations. The stream function, Ψ, in the hodograph plane is given by the well-known Chaplygin [3] equation:

$$(\partial^2\Psi/\partial t^2) + K(t)(\partial^2\Psi/\partial\theta^2) = 0. \tag{2.1}$$

Here, θ is the angle between the freestream direction and the velocity vector, and the Chaplygin variable, t, is defined as

$$t = \frac{1}{\rho_0}\int_{c_*}^{q} \frac{\rho}{q}\, dq, \tag{2.2}$$

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where ρ and q are the fluid density and the flow speed, respectively, c_* is the critical speed of sound, and the subscript 0 denotes the value at the stagnation point. Using the local Mach number, M, K is given by

$$K(t) = (1 - M^2)(\rho_0^2/\rho^2) = (1/Q)\, d^2Q/dt^2 \tag{2.3}$$

with $Q = q^{-1}$. The Liouville transformations

$$\tau = -\int_0^t K^{1/2}\, dt = -\int_{c*}^{q} (1 - M^2)^{1/2} \frac{dq}{q}, \qquad \psi^* = K^{1/4}\Psi, \tag{2.4}$$

reduce the Chaplygin equation (2.1) into the Bergman [2]-Imai [7] equation:

$$\partial^2\psi^*/\partial\tau^2 + \partial^2\psi^*/\partial\theta^2 = k(\tau)\psi^*, \tag{2.5}$$

with

$$k = K^{-1/4}\, d^2K^{1/4}/d\tau^2 = -K^{-3/4}\, d^2K^{-1/4}/dt^2. \tag{2.6}$$

After further transformations

$$\eta = (3\tau/2)^{2/3}, \qquad \psi = \eta^{-1/4}\psi^* = H(\eta)\Psi, \qquad H(\eta) = (K/\eta)^{1/4}, \tag{2.7}$$

we obtain the Diaz–Ludford [4] equation:

$$\partial^2\psi/\partial\eta^2 + \eta\, \partial^2\psi/\partial\theta^2 = h(\eta)\psi, \tag{2.8}$$

where

$$h(\eta) = (1/H)\, d^2H/d\eta^2 = k\eta + 5/(16\eta^2). \tag{2.9}$$

Substituting Eqs. (2.4) and (2.7) into Eq. (2.3), and using Eq. (2.9), we obtain

$$d^2y/d\eta^2 - (h + \eta)y = 0 \tag{2.10}$$

with $y = c_*QH/H(0)$. In a similar manner, from Eq. (2.2) and from the Bernoulli equation $q\, dq + \rho^{-1}\, dp = 0$, we get

$$\frac{\rho_0}{\rho} = \frac{H^2}{Q}\frac{dQ}{d\eta} = \frac{H^2}{y}\frac{dy}{d\eta} - \frac{1}{2}\frac{dH^2}{d\eta} \tag{2.11}$$

and

$$dp = (c_*^2\rho_0/c_1)\, d\eta/y^2, \tag{2.12}$$

respectively. Here, p is the pressure, and $c_1 = \{H(0)\}^2$. From Eqs. (2.4) and (2.7), we have

$$t = -\int_0^\eta H^{-2}\, d\eta. \tag{2.13}$$

3. First approximation for transonic flows. Near the sonic point, h is expressed as

$$h = \sum_{n=0}^{\infty} \lambda_n \eta^n. \tag{3.1}$$

Therefore, in the transonic range, h is approximately given by

$$h = \lambda^2 \tag{3.2}$$

with $\lambda^2 = \lambda_0$. Eq. (2.8) is reduced to

$$\partial^2\psi/\partial\eta^2 + \eta\, \partial^2\psi/\partial\theta^2 = \lambda^2\psi. \tag{3.3}$$

Transformations

$$\eta = \xi/\lambda, \qquad \theta = \lambda^{-3/2}\vartheta \tag{3.4}$$

simplify this equation into

$$\partial^2\psi/\partial\xi^2 + \xi\, \partial^2\psi/\partial\vartheta^2 = \psi. \tag{3.5}$$

It is remarked that Eq. (3.3) reduces to the Tricomi equation

$$\partial^2\psi/\partial\eta^2 + \eta\, \partial^2\psi/\partial\theta^2 = 0 \tag{3.6}$$

if $\lambda^2 = 0$. Such a treatment introduces a hypothetical gas, the generalized Tricomi gas. Our treatment introduces a hypothetical gas, which includes the generalized Tricomi gas as a special case, $\lambda = 0$. In Fig. 1, h is plotted versus η. The solid line is for the polytropic gas of $\gamma = 1.4$, γ being the ratio of specific heats, and the chain lines correspond to the first and the second approximations of the present paper; the broken line corresponds to the approximation developed by Diaz and Ludford [4].

In this transonic approximation, Eq. (2.9) reduces to $d^2H/d\xi^2 - H = 0$, whose solution satisfying the condition $H(0) = c_1^{1/2}$ is

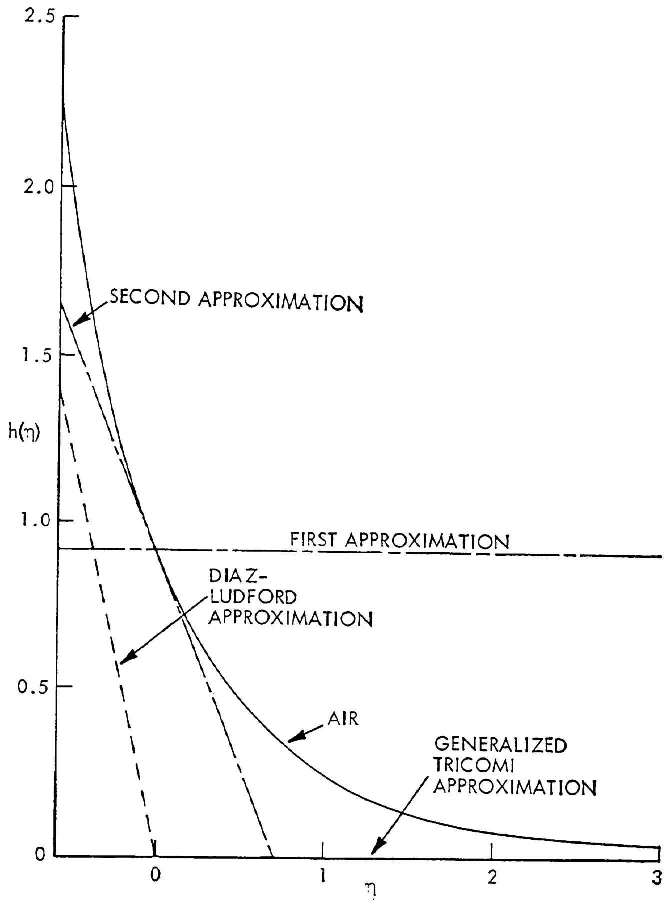

FIG. 1. The function $h(\eta)$ for the air and various approximations.

$$H = c_1^{1/2}\left(\cosh\xi - \frac{c_2}{\lambda}\sinh\xi\right), \tag{3.7}$$

c_2 being an integration constant.

From Eqs. (2.13), (3.4), and (3.7), we have

$$t = -(1/c_1)\sinh\xi/(\lambda\cosh\xi - c_2\sinh\xi). \tag{3.8}$$

Putting

$$\zeta = \eta + \lambda^2, \tag{3.9}$$

Eq. (2.10) is reduced to the Airy equation

$$d^2y/d\zeta^2 - \zeta y = 0. \tag{3.10}$$

The general solution of this equation may be written as

$$y = c_3\,\mathrm{Ai}\,(\zeta) + c_4\,\mathrm{Bi}\,(\zeta), \tag{3.11}$$

where c_3 and c_4 are integration constants, and Ai and Bi are the Airy functions[1] defined by

$$\mathrm{Ai}\,(\zeta) = \tfrac{1}{3}\zeta^{1/2}\{I_{-1/3}(\tfrac{2}{3}\zeta^{3/2}) - I_{1/3}(\tfrac{2}{3}\zeta^{3/2})\} = \frac{1}{\pi}(\zeta/3)^{1/2}K_{1/3}(\tfrac{2}{3}\zeta^{3/2}),$$

$$\mathrm{Bi}\,(\zeta) = (\zeta/3)^{1/2}\{I_{-1/3}(\tfrac{2}{3}\zeta^{3/2}) + I_{1/3}(\tfrac{2}{3}\zeta^{3/2})\},$$

$I_{\pm 1/3}$ being the modified Bessel functions of the first kind.

From (2.11), (3.4) and (3.9), we have

$$\frac{\rho_0}{\rho} = c_1\left\{\frac{c_3\,\mathrm{Ai}'\,(\zeta) + c_4\,\mathrm{Bi}'\,(\zeta)}{c_3\,\mathrm{Ai}\,(\zeta) + c_4\,\mathrm{Bi}\,(\zeta)}\left(\cosh\xi - \frac{c_2}{\lambda}\sinh\xi\right)^2 + (c_2\cosh\xi - \lambda\sinh\xi)\left(\cosh\xi - \frac{c_2}{\lambda}\sinh\xi\right)\right\}. \tag{3.12}$$

It is easy to show that $y\int_{\lambda^2}^{\zeta} y^{-2}\,d\zeta$ satisfies the Airy equation. Thus we may write

$$y\int_{\lambda^2}^{\zeta} y^{-2}\,d\zeta = c_5\,\mathrm{Ai}\,(\zeta) + c_6\,\mathrm{Bi}\,(\zeta),$$

where $c_5 = -\pi/(mc_3 + c_4)$, $c_6 = -mc_5$, $m = \mathrm{Ai}\,(\lambda^2)/\mathrm{Bi}\,(\lambda^2)$. Since Eq. (2.12) reduces to

$$dp = (c_*^2\rho_0/c_1)y^{-2}\,d\zeta,$$

after integration, we obtain

$$p = p_* - \frac{\pi c_*^2\rho_0}{c_1(mc_3 + c_4)}\,\frac{\mathrm{Ai}\,(\zeta) - m\,\mathrm{Bi}\,(\zeta)}{c_3\,\mathrm{Ai}\,(\zeta) + c_4\,\mathrm{Bi}\,(\zeta)}. \tag{3.13}$$

At the sonic point, $t = \tau = \eta = \xi = 0$ and $\zeta = \lambda^2$. Assuming that Eqs. (3.11) and (3.12) are valid here, we have

$$c_3 = \{\mathrm{Ai}\,(\lambda^2)\}^{-1} - m^{-1}c_4, \qquad c_4 = \pi\{c\,\mathrm{Ai}\,(\lambda^2) - \mathrm{Ai}'\,(\lambda^2)\} \tag{3.14}$$

with $c = -c_2 + \rho_0/(c_1\rho_*)$.

It is noted that $\rho \to 0$ as $M \to 0$ unless $\lambda/c_2 < 1$, in which case $\rho \to \infty$ as $\eta \to \lambda^{-1}\tanh^{-1}(\lambda/c_2)$.

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[1] For the formulas of the Airy functions, see Abramowitz and Stegun [1]. It is noted that their formula 10.4.16 for Ai′ actually gives −Ai′.

Now, if K is expanded at the sonic point as

$$K = a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots , \tag{3.15}$$

it is easily shown that

$$a_1 = -c_1^3 , \qquad a_2 = -5c_1^4 c_2 , \qquad a_3 = -c_1^5\{15c_2^2 + (7/3)\lambda^2\}. \tag{3.16}$$

Thus

$$\lambda^2 = (3/35)(5a_1a_3 - 3a_2^2)/(-a_1)^{8/3}, \tag{3.17}$$

which agrees with the result of the WKBI method obtained by Imai [6]. For the polytropic gas, Diaz and Ludford [4] show that

$$\begin{aligned} a_1 &= -(\gamma + 1)(\rho_0/\rho_*)^3, \qquad a_2 = -(\gamma + 1)(\gamma + 5/2)(\rho_0/\rho_*)^4, \\ a_3 &= -(\gamma + 1)\,\frac{6\gamma^2 + 25\gamma + 31}{6}\left(\frac{\rho_0}{\rho_*}\right)^5. \end{aligned} \tag{3.18}$$

If these values are substituted into Eqs. (3.16) and (3.17), we obtain

$$c_1 = (\rho_0/\rho_*)(\gamma + 1)^{1/3}, \qquad c_2 = (2\gamma + 5)/\{10(\gamma + 1)^{1/3}\}, \tag{3.19}$$

and

$$\lambda^2 = (24\gamma^2 + 70\gamma + 85)/\{140(\gamma + 1)^{2/3}\}. \tag{3.20}$$

For the air, $\gamma = 1.4$ and $\lambda^2 = 0.916645$.

In some of the practical applications, closer approximation in the subsonic range might be needed. Substituting Eqs. (3.19) into Eq. (3.13), and using the relation

$$c_*^2 \rho_*/p_* = \gamma, \tag{3.21}$$

we have $p/p_* \to 1 + \pi\gamma/\{(\gamma + 1)^{1/3}c_4(c_3 + m^{-1}c_4)\}$ as $\zeta \to \infty$. If this value is supposed to be equal to p_r/p_*, using Eqs. (3.14) and (3.19), we obtain

$$\frac{\mathrm{Ai}'(\lambda^2)}{\mathrm{Ai}(\lambda^2)} = \frac{1}{(\gamma + 1)^{1/3}}\left\{\frac{5 - 2\gamma}{10} - \frac{\gamma}{(p_r/p_*) - 1}\right\}. \tag{3.22}$$

For the polytropic gas

$$p_0/p_* = \{(\gamma + 1)/2\}^{\gamma/(\gamma-1)}. \tag{3.23}$$

Assuming that $p_r = p_0$ and putting $\gamma = 1.4$, the right-hand side of Eq. (3.22) becomes -1.0068; thus λ^2 is obtained as $\lambda^2 = 0.5826$.

Relations between ρ/ρ_* and $q_* = q/c_*$ are shown in Fig. 2 by the chain lines for $\lambda^2 = 0.5826$, 0.7, and 0.916645; also included in this figure are relations for the air (solid line), and (broken lines) for the generalized Tricomi gas, $\lambda^2 = 0$, and for the Tricomi gas. In Fig. 3, curves of p/p_* versus ρ/ρ_* are shown in a similar manner. Tierney [8] shows that, if two functions $K(t)$ have the same expansion in positive integral powers of t up to and including the term in t^n, the graphs in the (p, ρ)-plane of the corresponding equations of state have contact of order at least $n + 1$ at the sonic point (p_*, ρ_*). Thus, for the Tricomi gas and for the generalized Tricomi gas, contacts of second and third order, respectively, are obtained. In the present approximation, if the value of λ^2 given by Eq. (3.20) is used, contact of fourth order is achieved. Thus the chain line marked as $\lambda^2 = 0.916645$ has fourth-order contact to the solid line, giving a very good approximation in the transonic range. For the subsonic applications, a choice of $\lambda^2 = 0.7$ or 0.5826 might give better results.

4. Second approximation for transonic flows. In this section, the second term of the h series, Eq. (3.1), is taken into consideration, namely

$$h = \lambda^2 - \mu\eta \tag{4.1}$$

with $\mu = -\lambda_1 > 0$. Eq. (2.8) becomes

$$\partial^2\psi/\partial\eta^2 + \eta\, \partial^2\psi/\partial\theta^2 = -(\mu\eta - \lambda^2)\psi. \tag{4.2}$$

After transformations $\xi = (\mu/\lambda^2)\eta - 1$, $\vartheta = (\mu/\lambda^2)^{3/2}\theta$, we obtain

$$\partial^2\psi/\partial\xi^2 + (\xi + 1)\, \partial^2\psi/\partial\vartheta^2 = -\, (\lambda^6/\mu^2)\xi\psi. \tag{4.3}$$

Equation (2.9) is reduced to the Airy equation $d^2H/dx^2 - xH = 0$, where

$$x = -\mu^{1/3}(\eta - \lambda^2\mu^{-1}). \tag{4.4}$$

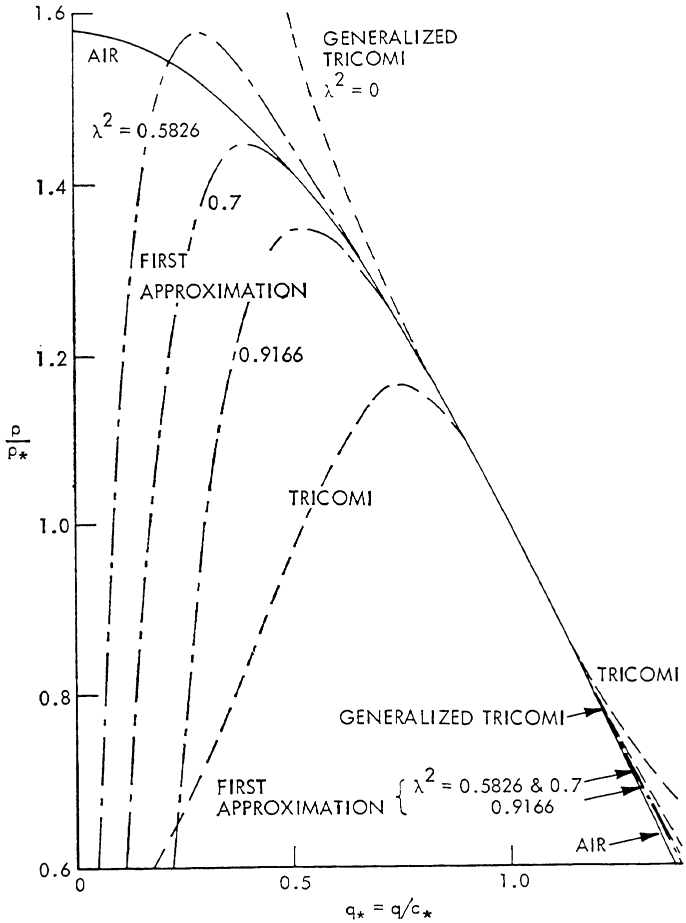

FIG. 2. The density-velocity relation by the first approximation.

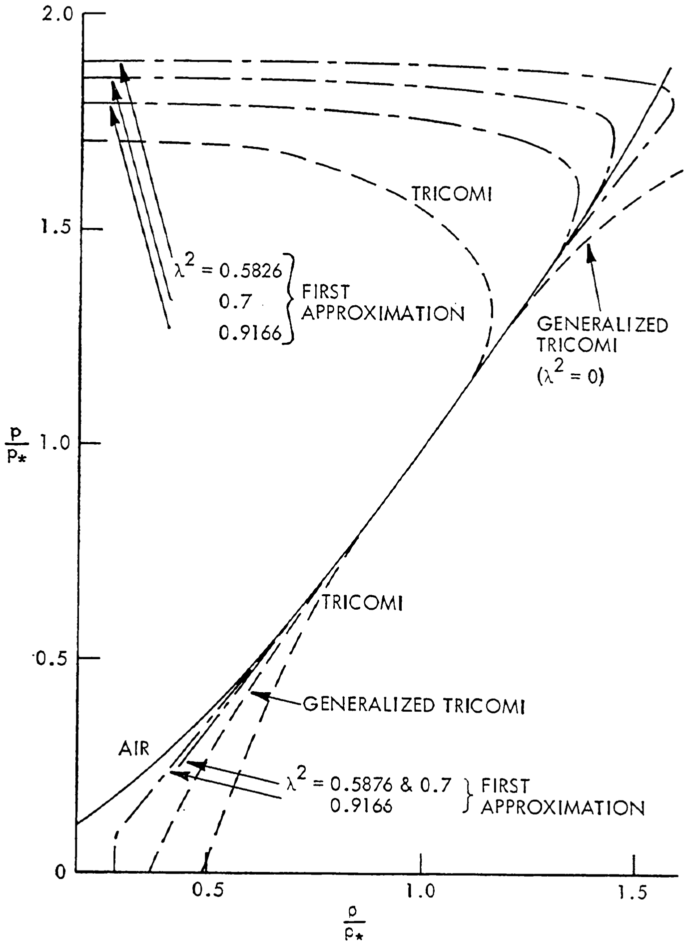

FIG. 3. The pressure-density relation by the first approximation.

Therefore, H may be written as

$$H = a \operatorname{Ai}(x) + b \operatorname{Bi}(x), \tag{4.5}$$

where a and b are determined by the conditions $H = c_1^{1/2}$ and $dH/d\eta = -c_1^{1/2}c_2$ at $\eta = 0$ as

$$a = c_1^{1/2}\pi\{\operatorname{Bi}'(\beta^2) - c_2\mu^{-1/3}\operatorname{Bi}(\beta^2)\},$$

$$b = c_1^{1/2}\pi\{c_2\mu^{-1/3}\operatorname{Ai}(\beta^2) - \operatorname{Ai}'(\beta^2)\},$$

$$\beta = \lambda\mu^{-1/3}.$$

It is noted that Eq. (4.5) tends to Eq. (3.7) asymptotically as $\mu \to 0$.

From Eqs. (2.13) and (4.4) we have $t = \mu^{-1/3} \int_{\beta^2}^{x} H^{-2}\, dx$. Since $H \int_{\beta^2}^{x} H^{-2}\, dx$ satisfies the Airy equation, we obtain

$$H \int_{\beta^2}^{x} H^{-2}\, dx = c_5 \operatorname{Ai}(x) + c_6 \operatorname{Bi}(x),$$

where $c_5 = -\pi/(na + b)$, $c_6 = -nc_5$, $n = \mathrm{Ai}(\beta^2)/\mathrm{Bi}(\beta^2)$. Thus

$$t = -\frac{\pi}{\mu^{1/3}(na + b)} \frac{\mathrm{Ai}(x) - n\,\mathrm{Bi}(x)}{a\,\mathrm{Ai}(x) + b\,\mathrm{Bi}(x)}. \tag{4.6}$$

For the case $\mu \neq 1$, putting

$$\zeta = (1 - \mu)^{1/3}\{\eta + \lambda^2/(1 - \mu)\},$$

we again obtain Eqs. (3.10) and (3.11) for y. Eq. (2.11) becomes

$$\frac{\rho_0}{\rho} = \{a\,\mathrm{Ai}(x) + b\,\mathrm{Bi}(x)\}^2 \times \left[\frac{c_3\,\mathrm{Ai}'(\zeta) + c_4\,\mathrm{Bi}'(\zeta)}{c_3\,\mathrm{Ai}(\zeta) + c_4\,\mathrm{Bi}(\zeta)}(1 - \mu)^{1/3} + \mu^{1/3}\frac{a\,\mathrm{Ai}'(x) + b\,\mathrm{Bi}'(x)}{a\,\mathrm{Ai}(x) + b\,\mathrm{Bi}(x)}\right]. \tag{4.7}$$

From Eq. (2.12), we have

$$dp = [c_*^2\rho_0/\{c_1(1 - \mu)^{1/3}\}]\, d\zeta/y^2.$$

Now we can write

$$y\int_{\zeta_0}^{\zeta} y^{-2}\, d\zeta = c_7\,\mathrm{Ai}(\zeta) + c_8\,\mathrm{Bi}(\zeta),$$

where $c_7 = -\pi/(m^*c_3 + c_4)$, $c_8 = -m^*c_7$, $m^* = \mathrm{Ai}(\zeta_0)/\mathrm{Bi}(\zeta_0)$, $\zeta_0 = \lambda^2/(1 - \mu)^{2/3}$. Therefore, p is obtained as

$$p = p_* - \frac{\pi c_*^2\rho_0}{c_1(1 - \mu)^{1/3}(m^*c_3 + c_4)} \frac{\mathrm{Ai}(\zeta) - m^*\,\mathrm{Bi}(\zeta)}{c_3\,\mathrm{Ai}(\zeta) + c_4\,\mathrm{Bi}(\zeta)}. \tag{4.8}$$

Assuming that Eqs. (3.11) and (4.7) are valid at sonic point, where $\zeta = \zeta_0$ and $x = \beta^2$, leads to

$$\begin{aligned} c_3 &= \pi\left\{\mathrm{Bi}'(\zeta_0) - \frac{c}{(1 - \mu)^{1/3}}\mathrm{Bi}(\zeta_0)\right\} = \frac{1}{\mathrm{Ai}(\zeta_0)} - \frac{c_4}{m^*}, \\ c_4 &= \pi\left\{\frac{c}{(1 - \mu)^{1/3}}\mathrm{Ai}(\zeta_0) - \mathrm{Ai}'(\zeta_0)\right\}. \end{aligned} \tag{4.9}$$

For the case $\mu = 1$, Eq. (2.10) reduces to $d^2y/d\eta^2 - \lambda^2 y = 0$, whose general solution may be written as

$$y = c_3 \cosh \lambda\eta + c_4 \sinh \lambda\eta. \tag{4.10}$$

From Eq. (2.11) we obtain

$$\frac{\rho_0}{\rho} = \{a\,\mathrm{Ai}(x) + b\,\mathrm{Bi}(x)\}^2\left\{\lambda\frac{c_3 \sinh \lambda\eta + c_4 \cosh \lambda\eta}{c_3 \cosh \lambda\eta + c_4 \sinh \lambda\eta} + \mu^{1/3}\frac{a\,\mathrm{Ai}'(x) + b\,\mathrm{Bi}'(x)}{a\,\mathrm{Ai}(x) + b\,\mathrm{Bi}(x)}\right\}. \tag{4.11}$$

Now it is easy to show that $y \int_0^\eta y^{-2}\, d\eta = (c_3\lambda)^{-1} \sinh \lambda\eta$. It follows that

$$p = p_* + \frac{c_*^2\rho_0}{c_1c_3\lambda} \frac{\sinh \lambda\eta}{c_3 \cosh \lambda\eta + c_4 \sinh \lambda\eta}. \tag{4.12}$$

The requirements that Eqs. (4.10) and (4.11) hold true at the sonic point, $\eta = 0$, determine that $c_3 = 1$ and $c_4 = c/\lambda$. It is to be noted that results for the case $\mu = 1$ can be derived from that for the case $\mu \neq 1$ by means of the asymptotic behaviors of the Airy functions.

Now the coefficient a_4 appearing in Eq. (3.15) is

$$a_4 = (c_1^6/12)(196\lambda^2 c_2 - 9\mu + 420c_2^3). \tag{4.13}$$

Thus

$$\mu = -(4/75)a_1^{-4}(14a_2^3 - 35a_1a_2a_3 + 25a_1^2a_4), \tag{4.14}$$

which again agrees with the results of the WKBI method. For the polytropic gas, it can be shown that

$$a_4 = -(\gamma + 1)\frac{24\gamma^3 + 134\gamma^2 + 283\gamma + 233}{24}\left(\frac{\rho_0}{\rho_*}\right)^6. \tag{4.15}$$

If this equation as well as Eqs. (3.18) are substituted into Eq. (4.14), we have

$$\mu = (32\gamma^3 + 90\gamma^2 + 95\gamma + 75)/\{150(\gamma + 1)\}. \tag{4.16}$$

For the air, $\gamma = 1.4$ and $\mu = 1.311689$.

In the application for subsonic and transonic range, it is possible to select the value of μ in such a way that $p/p_* \to p_0/p_*$ as $\eta \to \infty$. Since numerical experiments show that such μ is less than unity, we obtain, by substituting Eqs. (3.19) and (3.21) into Eq. (4.8),

$$p/p_* \to 1 + (\pi\gamma)/\{(\gamma + 1)^{1/3}(1 - \mu)^{1/3}c_4(c_3 + m^{*-1}c_4)\}$$

as $\eta \to \infty$ and $\zeta \to \infty$. Assuming that the value of the right-hand side is equal to p_0/p_*, and using Eqs. (3.19) and (4.9), we have

$$\frac{5 - 2\gamma}{10} - (\gamma + 1)^{1/3}(1 - \mu)^{1/3}\frac{\mathrm{Ai}'(\zeta_0)}{\mathrm{Ai}(\zeta_0)} = \frac{\gamma}{(p_0/p_*) - 1}.$$

For the polytropic gas of $\gamma = 1.4$, using Eq. (3.23), the right-hand side of this equation is calculated as 1.5678. Thus, μ is determined as $\mu = 0.81$.

Relations between ρ/ρ_* and q_* are shown in Fig. 4 by the chain lines for $\mu = 0.81$, 1.0, and 1.311689. Also included in the figure are the relation for the air (solid line) and curves for the first approximation, $\mu = 0$ and $\lambda^2 = 0.916645$, for the generalized Tricomi gas and for the Tricomi gas (broken lines). In Fig. 5, curves of p/p_* versus ρ/ρ_* are shown in a similar manner. By the Tierney result cited in the previous section, contact of fifth order is achieved in the second approximation, if the value of μ given by Eq. (4.16) is used. It also will be seen that the selection of $\mu = 1.0$ or 0.81 gives better results in the subsonic range.

5. Solutions for the stream function. In this section, solutions for the stream function are derived. The equation for the reduced stream function is

$$\partial^2\psi/\partial\eta^2 + \eta(\partial^2\psi/\partial\theta^2) = -(\mu\eta - \lambda^2)\psi. \tag{5.1}$$

Putting $\psi = Y(\eta)\Theta(\theta)$, we obtain

$$\Theta(\theta) = A_n \sin n\theta + B_n \cos n\theta, \tag{5.2}$$

A_n and B_n being integration constants, and $d^2Y/d\eta^2 - (m^2\eta + \lambda^2)Y = 0$ with $m^2 = n^2 - \mu$. If $m \neq 0$, using a new variable $\xi = m^{2/3}(\eta + \lambda^2 m^{-2})$, this equation is reduced to the Airy equation $d^2Y/d\xi^2 - \xi Y = 0$. Thus, $Y(\xi)$ may be written as

$$Y(\xi) = C_n \operatorname{Ai}(\xi) + D_n \operatorname{Bi}(\xi), \tag{5.3}$$

where C_n and D_n are integration constants. If $m = 0$, we obtain $n = \pm\mu^{1/2}$ and

$$Y(\eta) = C_n \cosh \lambda\eta + D_n \sinh \lambda\eta. \tag{5.4}$$

Thus, using Eqs. (2.7), (3.4), (3.7), and (4.5), we may write

$$\Psi = \Sigma C_n A^* e^{\pm in\theta}/[c_1^{1/2}\{\cosh \lambda\eta - (c_2/\lambda) \sinh \lambda\eta\}] \tag{5.5}$$

for the first approximation, and

$$\Psi = \Sigma C_n A^* e^{\pm in\theta}/\{a \operatorname{Ai}(x) + b \operatorname{Bi}(x)\} \tag{5.6}$$

for the second approximation. Here, A^* represents any appropriate linear combination of the Airy functions of ξ and the hyperbolic functions of $\lambda\eta$, and x is given by Eq. (4.4). It is noted that, for the Tricomi and the generalized Tricomi gases, $\lambda = \mu = 0$, we have

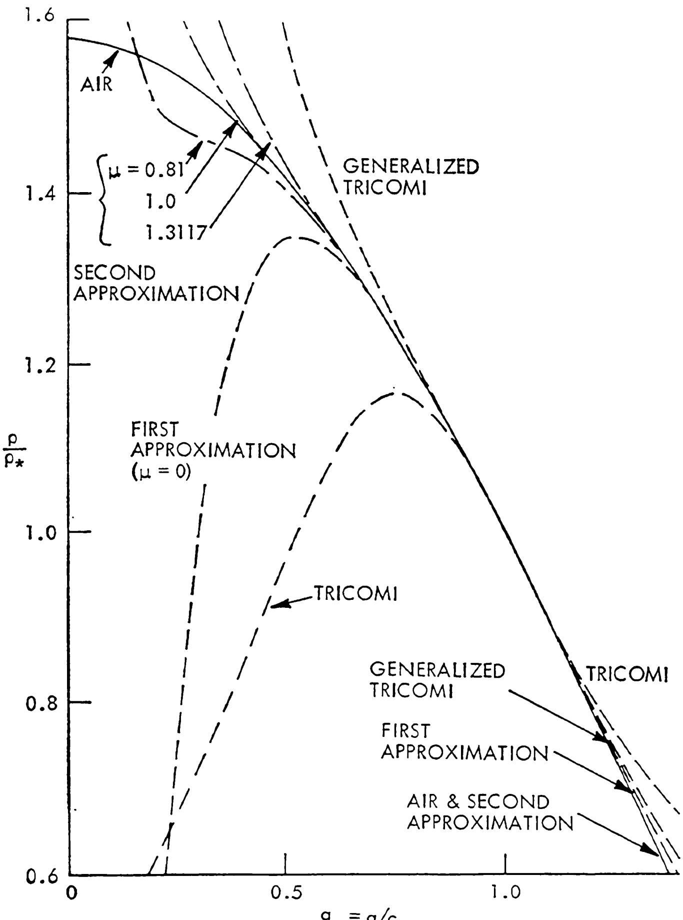

FIG. 4. The density-velocity relation by the second approximation.

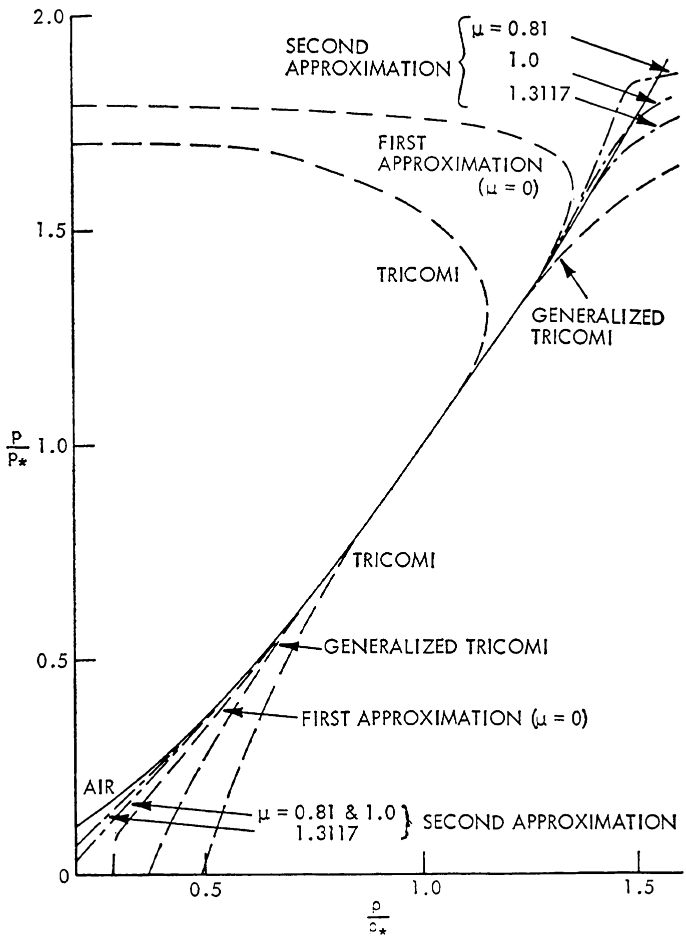

FIG. 5. The pressure-density relation by the second approximation.

$m = n$, so that $\xi = n^{2/3}\eta$; $H = c_1^{1/2}$ for the Tricomi gas, and $H = c_1^{1/2}(1 - c_2\eta)$ for the generalized Tricomi gas.

6. Concluding remarks. Using the formulas of the WKBI method, the first and the second approximations for transonic flows have been obtained, the zeroth approximation being the generalized Tricomi approximation. Applicable range of the approximation to the airflow is $0.65 < M < 1.4$ for the first approximation, and $0.5 < M < 1.5$ for the second approximation. This is compared to the corresponding ranges of $0.9 < M < 1.2$ for the Tricomi gas, and $0.75 < M < 1.3$ for the generalized Tricomi gas. Stream functions in these approximations are expressed by the Airy functions and the hyperbolic functions.

For practical applications where emphasis is placed in the subsonic range, it has been proved that, changing the values of integration constants, the applicable range of $0.45 < M < 1.35$ and $0.35 < M < 1.4$ can be obtained for the first and the second approximation, respectively.

For the second and higher approximations, equations for H and for y are of the same type. This is the special merit of this approximate method. For example, for the third approximation, where h is given by

$$h(\eta) = \lambda^2 - \mu\eta + \nu\eta^2,$$

both H and y can be expressed by the Weber functions. This suggests the possibility of the improvement of the WKBI method. Detailed results will be reported elsewhere.

References

[1] M. Abramowitz and I. A. Stegun (Editors), *Handbook of mathematical functions with formulas, graphs and mathematical tables*, 5th printing, with corrections, Nat. Bur. Standards, Appl. Math. Series, 55, U. S. Government Printing Office, Washington, D. C., 1966, pp. 446–450

[2] S. Bergman, *On two-dimensional flows of compressible fluids*, NACA Tech. Note No. 972, 1945

[3] S. Chaplygin, *Gas jets*, Scientific Memoirs, Moscow Univ., Math. Phys. Sec. **21**, 1–121 (1902); also NACA Tech. Memo. No. 1063, 1944

[4] J. B. Diaz and G. S. S. Ludford, *A transonic approximation*, Proc. Second U. S. Nat. Congr. Appl. Mech. (Ann Arbor, 1954), Amer. Soc. Mech. Engineers, New York, 1955, pp. 651–658

[5] C. Ferrari and F. G. Tricomi, *Transonic aerodynamics*, Academic Press, New York and London, 1968

[6] I. Imai, *On a refinement of the W. K. B. method*, Phys. Rev. **74**, 113 (1948)

[7] I. Imai, *Application of the W. K. B. method to the flow of a compressible fluid. I*, J. Math. Phys. **28**, 173–182 (1949)

[8] J. A. Tierney, *An approximation to transonic flow of a polytropic gas*, Amer. J. Math. **75**, 43–56 (1953)