QUARTERLY

OF

APPLIED MATHEMATICS

VOLUME XXIX | APRIL–JULY 1971 | NUMBERS 1–2

QUARTERLY OF APPLIED MATHEMATICS

The QUARTERLY prints original papers in applied mathematics which have an intimate connection with applications. It is expected that each paper will be of a high scientific standard; that the presentation will be of such character that the paper can be easily read by those to whom it would be of interest; and that the mathematical argument, judged by the standard of the field of application, will be of an advanced character.

Manuscripts (two copies) submitted for publication in the QUARTERLY OF APPLIED MATHEMATICS should be sent to the Editorial Office, Box F, Brown University, Providence, RI 02912, either directly or through any one of the Editors. The final decision on acceptance of a manuscript for publication is made by the Managing Editor. In accordance with their general policy, the Editors welcome particularly contributions which will be of interest both to mathematicians and to scientists or engineers. Authors will receive galley proof only. The author's institution will be requested to pay a publication charge of $30 per page which, if honored, entitles the author to 100 free reprints. Detailed instructions will be sent with galley proofs.

The current subscription price per volume (March through December) is $55. Single issues can be purchased, as far as they are available, at $14 and back volumes at $50 per volume. Subscriptions and orders for back volumes must be addressed to the American Mathematical Society, P.O. Box 1571, Providence, RI 02901-1571. All orders must be accompanied by payment. Other subscription correspondence should be addressed to the American Mathematical Society, P.O. Box 6248, Providence, RI 02940-6248. *Quarterly of Applied Mathematics* (ISSN 0033-569X) is published four times a year (March, June, September, and December) by Brown University, Division of Applied Mathematics, 182 George Street, Providence, RI 02912. Second-class postage paid at Providence, RI. POSTMASTER: Send address changes to *Quarterly of Applied Mathematics*, Membership and Sales Department, American Mathematical Society, Post Office Box 6248, Providence, RI 02940-6248.

Second-class postage paid at Providence, Rhode Island.
Publication number 808680 (ISSN 0033-569X).
10 9 8 7 6 5 4 3 2   95 94 93 92 91

—BOOK REVIEW SECTION—

Production theory and indivisible commodities. By Charles R. Frank, Jr. Princeton University Press, Princeton, N. J., 1969. xii + 141 pp. $6.50.

This book applies linear and integer linear programming to allocation problems in the firm in an effort to obtain economically meaningful results which would illuminate the thorny problem of allocating indivisible resources. (Of these the knapsack problem is the best known and simplest type.) The economic problem is developed with admirable clarity. Postulates on production activities are developed along the lines of Koopmans and Gale: additivity, the admissibility of inaction, free disposal, and closure; divisibility is, of course, omitted. The author rightly criticizes various attempts in the economic literature to explain away indivisibility, that is, the integer character of certain resources. A distinction is drawn between increasing returns due to fixed factors and due to indivisibilities, only the latter being considered. The author shows that under these conditions efficient points can be characterized in terms of efficiency prices such that profit maximization leads to negative but almost zero profits. A deficit would be made up by subsidies or through discriminatory pricing. This leads to a new view of odd-lot and discriminatory pricing. Thus, a case emerges for allocating indivisible goods through the market when suitable efficiency prices can be found. However, the limitations of this procedure are not made explicit to the extent that might be wished. Thus one is left to wonder how, say, the quadratic assignment problem might be tackled with these methods. This work should give further impetus to research on duality in integer programming. To the mathematician it presents an interesting demonstration of the economist's way of thinking on integer programming problems.

M. J. BECKMANN (*Providence, R. I.*)

Topics in dynamics. I. Flows. By Edward Nelson. Princeton University Press, Princeton, N. J., 1970. i + 118 pp. $3.00.

In his introduction the author states

"In classical mechanics the state of a physical system is represented by a point in a differentiable manifold M and the dynamical variables by real functions on M. In quantum mechanics the states are given by rays in a Hilbert space $\mathcal{H}$ and the dynamical variables by self-adjoint operators on $\mathcal{H}$. In both cases motion is represented by a flow; that is, a one-parameter group of automorphisms of the underlying structure (diffeomorphisms or unitary operators).

"The infinitesimal description of motion is in the classical case by means of a vector field and in quantum mechanics by means of a self-adjoint operator. One of the central problems of dynamics is the integration of the equations of motion to obtain the flow, given the infinitesimal description of the flow." The author then proceeds to develop the mathematical formalism needed for the study of flows. Chapter headings are as follows: 1. Differential calculus, 2. Picard's method, 3. The local structure of vector fields, 4. Sums and lie products of vector fields, 5. Self-adjoint operators on Hilbert space, 6. Commutative multiplicity theory, 7. Extensions of Hermitean operators, 8. Sums and Lie products of self-adjoint operators.

ROBERT EASTON (*Providence, R. I.*)

Stochastically dependent equations: an introductory test for econometricians. By P. R. Fisk. Griffin's Statistical Monographs and Courses, M. G. Kendall, Editor. Hafner Publishing Co., New York, 1967. 181 pages. $6.40.

This is a textbook written in a clear and straightforward fashion; it limits itself to the essentials and thus offers an excellent introduction for anyone with some statistical background to the peculiar problems of econometric estimation. These are concerned with simultaneous equations which are stochastically interdependent. A simple expenditure model is used to demonstrate that techniques beyond ordinary least-squares regression are required in order to obtain consistent estimates. The Cowles Commission methods are developed in three chapters, and this is followed by a discussion of k-class estimators along the lines of Theil et al. The chapter on measures of correlation, besides being of interest in its own right, leads to a brief discussion of test statistics. The complications arising out of autocorrelation—and in particular correlation between predetermined variables and disturbances—are briefly treated in a concluding chapter. The usefulness of the book for teaching and study is greatly enhanced by the numerical example of a three-equation model with three endogenous and three exogenous variables to which six alternative estimation methods are applied. Mathematical points are presented in appendices. This book opens indeed an economical way to the study of econometrics, and can thus be highly recommended.

M. J. BECKMANN (*Providence, R. I.*)

System theory. By L. A. Zadeh and E. Polak. McGraw-Hill Book Co., New York, 1969. xiii + 521 pp. $18.50.

There are usually several drawbacks to a book consisting of self-contained chapters written by different authors, in terms of non-uniform terminology as well as fundamentally different approaches to the subject. In the present case the multitude of authors is, however, a decided advantage since the field of system theory has not yet advanced to the stage where a generally accepted approach is available.

The book is a collection of thirteen chapters written by twelve authors. It is divided into five parts: general system theory, linear systems, nonlinear systems, stochastic and learning systems and optimal systems.

The introductory chapter gives Professor L. A. Zadeh's approach to the foundations of general system theory, covering basic notions such as systems state and aggregate. It is the most readable account of Zadeh's ideas the reviewer has seen. The reader should, however, observe that the foundations of system theory is still in a state of flux and that there is not yet any generally accepted way to formalize the fundamental concepts of the theory. Different approaches have been suggested by R. E. Kalman and T. Windeknecht. References to these works are given in Zadeh's chapter. The first part also contains an account of finite-state systems by A. Gill, a presentation of linear time-varying systems by A. V. Balakrishnan and an application to network problems by T. E. Stern. Balakrishnan's treatment of time-varying systems is in essence an exposition based on his papers in the field. It is very general because he makes no assumptions on the dimensionality of the state psace. He also permits highly irregular systems since he allows the outputs to be more irregular than the inputs.

The second part of the book covers linear systems with both finite and continuous state spaces. It also includes a very readable account of Kron's method of "tearing" written by K. B. Harrison.

The part on nonlinear systems covers stability theory. A. Letov has written a chapter on Liapunov theory and I. W. Sandberg presents recently developed function space techniques for analysis of input-output stability.

Stochastic systems are represented by a chapter on stochastic finite state systems written by J. W. Carlyle and one chapter on learning systems by K. S. Fu.

The last part of the book deals with decomposition of large scale systems and contains a very nice exposition of very general conditions for optimality.

The book gives a good picture of some important aspects of system theory. Both material and authors have been chosen with great care. The different chapters are written by authors who are active researchers in their respective fields. The book requires a reasonable mathematical maturity. The reader is thus expected to know algebra, differential equations, functional analysis and probability theory. The book can be highly recommended to all persons wishing to know more about system theory, researchers in industrial laboratories, students and teachers.

KARL J. ÅSTRÖM (*Providence, R. I*)

Einstein spaces. By A. Z. Petrov. Pergamon Press, Oxford, 1969. xiii + 411 pp. $12.00.

This book was first published in Russian in 1961. It was translated into German in 1964 and now appears in an English translation made by R. F. Kelleher and edited by J. Woodrow, with amendments and revisions supplied by the author. The title is somewhat deceptive. Petrov defines an Einstein space as a Riemannian manifold in which the Ricci tensor is proportional to the metric tensor, but (as he points out in the Preface) the scope of the book is actually very much wider than that. It deals with Riemannian geometry in general, with special emphasis on the four-dimensional space-time of general relativity.

There are nine chapters: 1. Basic tensor analysis; 2. Einstein spaces; 3. General classification of gravitational fields; 4. Motions in empty space; 5. Classification of general gravitational fields by groups of motions; 6. Conformal mapping of Einstein spaces; 7. Geodesic mapping of gravitational fields; 8. The Cauchy problem for the Einstein field equations; 9. Special types of gravitational fields. There are many exercises and examples throughout the book, which ends with a bibliography containing 509 references and an index which, for a book of this type, should be much more detailed.

The author is Professor of Relativity and Gravitation at Kazan University, and his name is familiar to relativists chiefly in connection with his work on the classification of space-times through the algebraic properties of the Riemann tensor, or of the Weyl tensor. This is dealt with in the book under review, but it is only a part of a much wider panorama. For many years Eisenhart's *Riemannian geometry* has been the standard text to which relativists have turned for the basic mathematics of their craft. It seems to me that Petrov's book will now take over that role.

The notation is elegant and carefully devised. It is inevitably complicated, but the printer has done a good job and any misprints I have noticed are rather trivial: thus, for example, there is a prime missing in formula (1.13), in line 1 of p. 11 the subscripts of the second S should be enclosed in parentheses, and near the middle of p. 81 two superscripts should be subscripts. This is a book for the mathematically sophisticated reader and he should have no difficulty in putting such details right for himself. He may find it harder to accept the statement (pp. 21, 22) that, in a given Riemannian space, a tensor field cannot be given arbitrarily, or the appearance in equation (6.2) of the covariant derivative of a vector field given along a curve. But these are very minor points of criticism indeed when viewed against the whole great design.

J. L. Synge (*Dublin*)

Elasticité linéaire. By L. Solomon. Masson et Cie, Paris, 1968. xix + 742 pp. 150 F.

After discussing in the "avant-propos" the mathematical approach in the elasticity and some historical information about the development of this theory, the author describes the "model of an elastic body". He formulates in a rigorous form the hypotheses which he made about the homogeneous and isotropic medium to be considered in the following.

The book consists of ten chapters: the displacement and deformations of a continuous medium, the tensions, connections between the tensions and deformations, the equations of linear elasticity, the one-dimensional case, the two-dimensional case, the three-dimensional case, sphere, half-space and elastic contact. The theory of analytic functions, including conformal mapping and its application, is presented in the Appendix.

In the opinion of this reviewer, this book is an excellent presentation of the theory and should be consulted by everybody working in the mathematical theory of elasticity. The author compares his hypotheses with the physical reality. Then he deduces in a rigorous way from the hypotheses made in the first chapters the partial differential equations of the elastic problems and discusses the solutions. A very extensive bibliography is added (mostly indicating literature of eastern European countries).

The reviewer has the impression that some of the results could be generalized and improved by using orthogonal functions. An elastic vector can be developed in a bounded domain B of finite connectivity in a series of vectors which are orthogonal in B (see, e.g., Math. Ann. 98, 1927, pp. 248–263).

S. Bergman (*Stanford*)

Mathematische Hilfsmittel des Ingenieurs, Volume II. Edited by R. Sauer and I. Szabó. Springer-Verlag, Berlin, Heidelberg, New York, 1969. xx + 684 pp. $37.50.

Volume II of this work (Volumes I and III have already appeared) is particularly interesting because its treatment of ordinary and partial differential equations is arranged differently from that of most books. It is subdivided into two major parts: D: Initial value problems of ordinary *and* partial differential equations (by W. Törnig) and E: Boundary and eigenvalue problems of ordinary *and* partial differential equations and integral equations (by L. Collatz and R. Nicolovius). Most readers would expect a subdivision into 1) ordinary and 2) partial differential equations. The grouping here has the advantage that obtaining solutions by numerical methods can be more easily discussed. Part D contains partial differential equations of hyperbolic and parabolic type and part E those of elliptic type and the treatment of integral equations.

One is at once impressed by the feeling that the authors not only know the subject matter very well but also have great practical experience. Indeed, the reader will find either at the beginning or at the end of each longish section a compact discussion of why certain procedures should take second place compared to others in the solution of practical problems. Particular attention should be drawn to some examples which show how a direct numerical attack without knowledge of basic analytic theorems can lead to solutions which are erroneous or not unique.

I. Flügge-Lotz (*Stanford*)

Statistics in endocrinology. Edited by Jane W. McArthur and Theodore Colton. M.I.T. Press, Cambridge, Mass., 1970. xii + 476 pp. $12.50.

This book is composed of the proceedings of a conference bringing together statisticians and endocrinologists. The book contains articles and discussion of the articles by conference participants.

The purpose of the conference and of the book was, as stated in the preface, to acquaint endocrinologists with "modern experimental design and statistical analysis". The book is intended to provide an introduction to a broad range of topics for biologists with a knowledge of the more elementary numerical techniques used in bioassay. It should fill that goal not only for endocrinologists, but also for pharmacologists, biochemists, physiologists, or psychologists with an interest in bioassay and associated statistical methods. It would not be my choice for a general reference work or a textbook. However, the scientist or physician whose interest is aroused by some topic in the book should have no difficulty finding suitable reference sources and texts in the ample bibliography.

The chapter titles summarize the contents well. These include: experimental design, analysis of variance, multiple comparisons, continuous response covariance analysis in bioassay, and non-parametric statistics. General principles of bioassay are ably discussed in several places. Most of the topics are illustrated with examples and data taken from problems in endocrinology. The discussions are good. They include elementary questions as well as sophisticated ones and should help the reader decide whether he wishes to study a particular topic further. While the articles on statistics present nearly the best this discipline has to offer endocrinologists, this is not the case in those articles described as examples of biomathematics. The collection of articles on competitive protein binding represent biomathematics more ably. The models are more interesting and their presentations more scholarly. In this section we are treated to a truly impressive combination of mathematical models and statistical procedures for using the models. Topics introduced include: some mathematical models for saturation assays and radio immunoassay, nonlinear regression techniques for fitting models, and problems of experimental design and interval estimation.

The reader cannot expect a complete education in any topic covered by the book, but should be a better judge of what education he will seek after reading it.

Louis Homer (*Providence, R. I.*)

Thermoelasticity: basic theory and applications. By A. D. Kovalenko. Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969. x + 215 pp. $11.00.

This book is composed of seven chapters, five of which are direct condensations of the corresponding treatments in Boley and Weiner, *Theory of Thermal Stresses*, J. Wiley and Sons, 1960. The condensed treatment is, in the reviewer's opinion, too drastic to be sufficiently accurate or clear; futhermore, it has not been brought up to date in the intervening decade. Chapters V (44 pages) and VI (28 pages) give discussions of new material on shells of revolution and on axisymmetric thermoelastic problems. No information is included on several important topics such as plastic and visoelastic behavior, uniqueness theorems, thermoelastic beam theory, large deflections of plates, thermal stresses in structural numbers, etc. An appendix by J. B. Alblas on thermoelastic stability of plates has been added.

B. A. BOLEY (*Ithaca, N. Y.*)

Carl Friedrich Gauss: a biography. By Tord Hall. Translated from the Swedish by Albert Froderberg. M.I.T. Press, Cambridge, 1970. 170 pp. $7.95.

This is a lightweight mathematical biography of Gauss. The main facts about his personal life are given (his son Eugene melted down one of his father's medals he had inherited to make eyeglass frames) but the bulk of the book is devoted to his scientific accomplishments. There are individual chapters on astronomy, calculus of probabilities, geodesics, non-Euclidean geometry, physics, function theory and number theory.

I enjoyed the book thoroughly and then passed it on to a college sophomore majoring in mathematics. His comment was revealing: "A reader who knows the mathematics will be bored. A reader who doesn't will be baffled". This shows how difficult it is for an author to pitch the level of a book "about" mathematics.

P. J. DAVIS (*Providence, R. I.*)

Kontinuumstheorie strömender Medien. By Heinz Schade. Springer-Verlag, Berlin, Heidelberg, New York, 1970. xxiv + 275 pp. $18.70.

This short book, consisting of a mere 224 pages of text (i.e. exclusive of the 45-page-long appendix), ambitiously undertakes to introduce the subject of continuum fluid dynamics to students of engineering, physics, and applied mathematics about half-way through their careers as undergraduates. In line with modern tendencies towards integration, its aim is to cover in a single series of lectures the material which, at least in German universities, is usually dispersed among courses in mechanics, thermodynamics, fluid mechanics, gas dynamics and electrodynamics. The spirit of the presentation is that represented by C. Truesdell's and R. Toupin's *The classical field theories* and J. Serrin's *Mathematical principles of classical fluid mechanics*, both of which appeared in the monumental *Handbuch der Physk* edited by S. Flügge.

In order to keep the presentation brief and to the point, the author has adopted the postulational method. According to the author's own statement: "Fundamental physical quantities are not explained in words and no indications are given on how to measure them; they are introduced by name and symbol with no further comment. Similarly, the fundamental physical laws are not abstracted from thought experiments or real physical observations, but are stated without justification." However, a departure from this principle is made in Section 2.4 (on electrodynamics) "because, speaking generally, electrodynamical quantities are less familiar than are concepts in geometry, kinematics, mechanics and thermodynamics".

The thermodynamic equations and formulations include the dissipation function, but the author steers clear of the concept of entropy production and does not mention the Onsager-Casimir relations. The Coleman-Noll-Truesdell principle of thermodynamically compatible determinism is also left unmentioned, the treatment being confined to materials without "memory", i.e. essentially to a formulation which is compatible with the principle of local state.

The fundamental, mathematical concept which is needed to fulfill this program is that of tensor calculus. It is, therefore, not surprising that 26% of the book (58 pages out of 224) is taken up by the study of this subject (both in Cartesian and symbolic notation). Even so, mathematical proofs which cannot be given in a couple of lines are omitted because "tensor calculus is treated as no more than a mathematical tool".

Naturally enough, the tight discipline which the author imposes on himself does not allow him to go very far in the treatment of applications. Essentially, these are confined to a derivation of the fundamental equations of: hydrostatics, hydrodynamics (Navier-Stokes and energy equation, Reynolds equations, Euler's equations, potential flow equations), gas dynamics (nonviscous flows only, but including acoustics) and magnetohydrodynamics (incompressible flows only). The total space devoted to applications is four pages short of that given to tensor calculus.

The narrative in the book flows with assurance and clarity, the latter being enhanced by the fact that the author goes out of his way to identify and to put right the mental steps which usually cause students to falter or to go astray.

A potential user's attitude towards the book will depend on his opinion concerning the pedagogical merits or demerits of the postulational method. There is not doubt about the fact that the postulational method results in brief and often incisive presentations, thus quickly opening the way to the subsequent mathematical manipulation of the resulting differential equations under a variety of special conditions. On the other hand, physical imagination fails to be developed and the student finds it very difficult himself to extend the fundamental laws to such complex flows as those involving the simultaneous effects of viscosity and compressibility, the presence of diffusion, chemical reactions, ionization and so on.

The book contains 49 worked problems, mostly of a mathematical nature.

JOSEPH KESTIN (*Providence, R. I.*)

Stationary stochastic processes. By T. Hida. Princeton University Press and the University of Tokyo Press, Princeton, 1970. 161 pp. $3.00.

This is a series of lectures delivered at Princeton in 1967–68. The first part begins in a very elementary way with the standard facts about the one-dimensional Brownian motion and more general additive processes. The next topic is the modern (nuclear space) machinery for stationary processes, as introduced in recent years by Gelfand-Villenkin [*Generalized Functions*, Vol. 4, Academic Press, 1961], including Fourier transforms of probability measures on the dual of a nuclear space. The rest of the book is occupied chiefly with the special case of (Gaussian) white noise, alias Brownian motion, which is the heart of the matter. Now the exposition becomes more detailed, covering the Gauss and Fourier-Wiener transforms, Fourier-Hermite polynomials alias multiple Wiener integrals, the polynomial chaos of N. Wiener, the flow Brownian motion (shift), the infinite-dimensional rotation group and the associated Laplacian, translation in path space and the Cameron-Martin formula, all done in a concise and very readable way.

This white noise material all stems from the pioneering work of Wiener in which the emphasis is on Brownian motion as the proper setting for infinite-dimensional calculus, instead of on the actual sample paths, as in the work of P. Lévy. The former point of view is now coming back into fashion after an undeserved period of neglect, and it is appropriate that the author has included an (all too brief) sketch of Wiener's ideas on the analysis and synthesis of non-linear networks by means of white noise. It may be said that Wiener's program was a failure. He wanted to use the white noise to make actual (path-wise) models of the action of non-linear devices and that is probably too hard except in the Gaussian (linear) case. But he has left us a largely unexploited framework in which to put such problems, and I think it may be time for another attack.

Hida's book is short, well-written, and cheap, providing a good introduction to the (modern) mathematical background to Wiener's work and to the problems that it leaves unanswered. For the actual substance of these problems, the reader will have to report to the master himself.

HENRY MCKEAN (*New York*)

Mathematics applied to physics. Edited by E. Roubine. Springer-Verlag New York, Inc., New York, Unesco, Paris, 1970. xix + 610 pp. $16.00.

This volume, devoted to mathematics that has applications to physics, is directed to readers with a good background in mathematics and physics "for brushing up or adding to their knowledge, modernizing their teaching or assisting them in research." This is manifestly a difficult task and has been carried out remarkably well. A specialist may be prone to find fault with the presentation of his field of interest. This reviewer, for example, feels that the least successful chapters are those on ordinary differential equations and optimization. They are really not bad although the bibliographies are hardly adequate. On the other hand there are excellent up-to-date presentations of the theory of distributions (de Jager), exterior differential forms (Deschamps), partial differential and integral equations (John), numerical approximative of solutions of partial differential equations (Lions), and probability theory (Welsh). There is a concise chapter on functions of complex variables (Sommer) and a brief introduction to group representations for quantuum mechanics (Yamanouchi). The chapters on optimization and numerical approximation are in French and the remainder of the book is in English. The book arises from Unesco's efforts to improve the teaching of science and in that direction the book makes a significant contribution. Although not a textbook, it can be recommended to advanced students, teachers and the general users of mathematics.

J. P. LaSalle (*Providence, R. I.*)

Lectures on boundary theory for Markov chains. By Kai Lai Chung. Annals of Mathematics Studies, Princeton University Press and the University of Tokyo Press, Princeton, N. J., 1970. xvi + 94 pp. $3.00.

A Markov chain is a stochastic process governed by a transition probability $p_{ij}(t)$ (the probability of a transition from the ith to jth state in time t.) It is supposed that $(p_{ij}(0))$ is the identity matrix and that $p_{ij}(0+) = p_{ij}(0)$. In simple cases, for example if there are only finitely many states, $q_{ij} = p'_{ij}(0)$ exists (finite), $\sum_j q_{ij} = 0$ and the process proceeds as follows. Starting at state i the process will remain at i for a time having distribution density $q_i e^{-q_i t}$ (where we write q_i for $-q_{ii}$), then jump to a second state j with probability q_{ij}/q_i, remaining there for a time having distribution density $q_j e^{-q_j t}$, will then jump to a third state k with probability q_{jk}/q_j and so on. If a state r is ever reached for which $q_r = 0$, the process stays at r forever; if no such state is reached the jump times become infinite. In either case the path is a step function whose discontinuities are isolated jumps. If the number of states is infinite, q_i may be infinite, the preceding description is no longer valid and the sample functions at first appear hopelessly messy. The study of the sample functions has gone in two directions. On the one hand elementary continuity properties of the sample functions were discovered, properties which became more understandable when it was seen that the countable state space could be immersed in a topological state space chosen in such a way that the sample functions have the usual properties of nice Markov process sample functions: right continuity and left limits. On the other hand, as in this book, the following more constructive approach has been taken. It is supposed that $q_i = \sum_{j \neq i} q_{ij} < \infty$, so that the sample function properties in the finite state space case are valid except that the sucessive jump times may have a finite supremum T, and the problem then becomes that of analyzing the process after time T. This has required analyzing the ways paths can 'go to infinity', that is the qualitatively distinct asymptotic characters of paths at $T-$. This analysis has been accomplished by 'boundary theory', the adjunction of ideal points to the state space. It is supposed, in order to get elegant results, that the set of adjoined points, the 'boundary', is countable, and the transitions from the original states to the boundary points and back, and the transitions between boundary points are analyzed. Chung's book elucidates fully and clearly this kind of boundary theory, much of it due to Chung himself, including the necessary background of Markov chain theory. The book will be useful to everyone interested in continuous-parameter Markov chains.

J. L. Doob (*Urbana, Ill.*)

Linear mathematics: an introduction to linear algebra and linear differential equations. By Fred Brauer, John A. Nohel, and Hans Schneider. W. A. Benjamin, Inc., New York, 1970. xiii + 347 pp. $13.95.

This book is a text designed to accompany a one-semester introductory course in linear algebra and the theory of linear ordinary differential equations. The authors have in mind a course to be taken by mathematics and science students at the sophomore or perhaps junior year level. It is assumed that these students have already had two or three semesters of calculus with perhaps a very brief introduction to the theory of ordinary differential equations.

The book is divided into seven chapters. In the first chapter the authors present eight problems from various areas of science and technology. Each of these problems involves either a system of linear algebraic equations in several unknowns or one or more linear ordinary differential equations. While discussing each problem the authors stress the phenomenon of linearity, and they indicate that the given problem will be solved as appropriate techniques are developed in the remaining six chapters of the book.

The second chapter concerns matrices. Addition and multiplication of matrices are defined and discussed. The notion of a non-singular matrix is introduced.

In Chapter 3 the authors take up the theory of systems of linear algebraic equations in several unknowns. The presentation is thorough. Essentially, the student is introduced to the method of Gaussian elimination and some related ideas. However, he is not yet introduced to the concept of a linear space or any of the notions consequent therefrom.

Chapter 4 concerns determinants. Here it is shown that an $n \times n$ matrix is non-singular if and only if its determinant is non-zero. The chapter also includes Cramer's Rule.

In Chapter 5 the student meets the concept of a linear space, and he learns about the notions of linear independence and dimension. These ideas are then used to develop further the theory presented in Chapter 3. The student learns the relation between the homogeneous and non-homogeneous linear equation in Euclidean n-space.

In Chapter 6 the authors present the theory of linear ordinary differential equations. They discuss the notions of a fundamental matrix solution and a Wronskian. They discuss the homogeneous equation with constant coefficients. The notions of eigenvalue and eigenvector for an $n \times n$ matrix are introduced. (The Jordan canonical form is discussed in an appendix.) They discuss the asymptotic behavior of solutions as $t \to \pm \infty$ particularly for systems in the plane.

Chapter 7 concerns the Laplace transform and its application to linear ordinary differential equations.

The presentation throughout the entire text is thorough, perhaps even painstaking. There are many examples and exercises, some of which are used to introduce new concepts.

I have only two criticisms of this book. The first pertains to Chapter 4. Here, the authors define the notion of an $n \times n$ determinant inductively on n using the expansion by cofactors of the first row. I think that they should have used the definition rendered in most modern algebra texts; namely, the definition in which the determinant is given as a certain polynomial of its own elements. Indeed, on the one hand, using this definition, one can develop the properties of determinants as easily as one can using the authors' definition. On the other hand, this is the normal definition given in most modern algebra courses, so why not use it?

My second criticism is more tentative than my first. I should like to see more practical problems interspersed throughout the text. By this I mean problems from science and technology. I think that in a course such as the one for which this book is intended students must acquire some ability to apply the mathematics they learn to scientific and practical situations. This will broaden their appreciation of the material being studied and help them to relate it to their respective fields of interest.

To conclude, I think that this book will be a useful text for the type of course for which it is intended. On the whole, I believe that the authors have echieved their stated goals in a better than satisfactory manner.

NATHANIEL CHAFEE (*Providence, R. I.*)

Elementary particle theory. By A. D. Martin and T. D. Spearman. North-Holland Publishing Company, Amsterdam and London and American Elsevier Publishing Company, New York, 1970. 527 pp. $27.50.

One natural way of classifying the books in the area of elementary particle physics might be to see which aspect between *the phenomenology* and *the formal theory* is given more emphasis by the authors. This new book by Martin and Spearman belongs in such a classification to the former and contains no discussions on quantum field theory except some trivial introductory remarks. This is due to the author's bias, admittedly, toward the S-matrix theory of the strong interactions, which in their opinion "has been most successful for describing strong interactions". Because of such biased preference, these authors center primarily on the S-matrix theory and other topics such as the weak and electromagnetic interactions are not given any space after some introductory remarks. The developments as well as applications of the SU(3) theory and current algebra are not treated either. Although a rather lengthy introduction on the theoretical approaches of the elementary particle interactions is made in Chapter 1 (p.22–p. 44) in an apparent attempt to reduce such a gap, since no introduction can describe the theories deep enough to teach how to calculate, the discussions on the weak and electromagnetic interactions and current algebra in this chapter end up with incomplete statements. (The explanation of the Adler self-consistency condition on p. 32 certainly needs improvement as the amplitude for πN scattering *minus the nucleon Born term* vanishes as one of the pion four-momenta vanishes).

However, the above remarks should not be interpreted as pejorative; the book has accomplished to a great extent the task that the authors have set up in the beginning and become a refreshingly unique S-matrix text for strong interactions treating an arbitrary spin situation. The book contains a good summary of the crossing and analyticity properties of the helicity amplitudes in Chapter 7, following to a large extent the work on the crossing relations by T. L. Treuman and G. C. Wick and another work on the kinematical singularities by T. L. Treuman, thus enabling the readers to understand quickly the up-to-date works in this area. There is a whole chapter of discussions of the Regge pole theory (Chapter 9) which includes in addition to the usual materials the Sommerfeld-Watson-Regge transformation of helicity amplitudes, the finite energy sum rules and even some brief introductions to the concept of duality and the Veneziano model. Also Chapter 8 is totally devoted to the properties of the partial-wave amplitudes, methods of solution for the partial-wave dispersion relations and some nice results on the asymptotic bounds of the scattering amplitudes derived by using unitarity and analyticity properties of the amplitudes. The materials in Chapters 2–6 are the usual things that can be found in other books. But the simple discussions of the inhomogeneous Lorentz group focussing on two-particle helicity states in Chapter 3 and the concise treatments of the dispersion relations for the amplitude in Chapter 6 are commendable.

In short, this book should be a useful text of the S-matrix theory of the strong interactions for the people who intend to carry out theoretical researches in strong interaction physics.

Kyungsik Kang (*Providence, R. I.*)

Principles of crystal structure determination. By Gene B. Carpenter. W. A. Benjamin, Inc., New York, 1969. xii + 237 pp. $14.50.

Until recently the chemistry or other science department wishing to teach a beginning course in crystal structure determination by X-ray diffraction was faced with something of a paucity of suitable textbooks. Fortunately, the situation appears to have been considerably relieved within the last year or two by the appearance of such books as those of Stout and Jensen, Carpenter, Woolfson, and Buerger.

Prof. Carpenter has written a book which is, on the whole, quite admirable. It sets forth everything that the student of the method is likely to need to know and at the same time manages, in Prof. Carpenter's own words, "to avoid fruitless rigor, unfamiliar mathematical methods, needless elaboration of favorite subjects, and subjects that are chiefly of historical interest". The writing is unusually clear and pleasant, the book is attractively printed and well proofread, and there is a selection of quite suitable problems at the end of each chapter. The book is clearly one that can be successfully taught from. There is a useful bibliography at the end.

There are certain slight faults in the book, in the reviewer's opinion. Curiously, not a single example of an electron-density map, which would illustrate what a finished structure determination might look like, is given. From this omission arises, perhaps, a slight air of unrealness about the final part of the book, which deals in two well-summarized but exceedingly brief chapters with the ways in which crystal structures are finally derived from the experimental data. On the other hand, three of the early chapters (4, 5, and 7), which deal with the opposite question—how the X-ray scattering depends upon the scattering structure—are unnecessarily long and unclear because the simple unifying concept of the X-ray experiment as a measurement of the Fourier transform of the structure (used by every working crystallographer today) is only introduced when the discussion is nearly at an end. The loss of insight which the student will experience by this treatment is perhaps the book's worst fault. At the same time Prof. Carpenter's technique of deferring the discussion of crystal symmetry until late in the book is a welcome and successful departure from the traditional method.

Finally, one perhaps minor point. On p. 188 Prof. Carpenter begins the discussion of the principal so-called direct methods for structure determination with a reference to a statistical relation among the structure factors (or Fourier coefficients of a structure) discovered by the reviewer. It is true that a relation which is of statistical validity only can be drawn from the equation which I gave, but that relation—that $F(\mathrm{S})$ tends to have the same phase as $F(\mathrm{S}')F(\mathrm{S} - \mathrm{S}')$ when all three terms are large—was given by others before me, while the equation itself (Eq. 11–30 in Prof. Carpenter's book) is both a more complete result and a more exact one, in the sense that it is non-statistical in nature. The point is perhaps a small one, but it may be well to emphasize that the direct methods are not based exclusively on statements of a probabilistic nature.

DAVID SAYRE (*Yorktown Heights, N. Y.*)