QUARTERLY

OF

APPLIED MATHEMATICS

VOLUME XXVIII OCTOBER • 1970 NUMBER 3

QUARTERLY OF APPLIED MATHEMATICS

This periodical is published quarterly by Brown University, Providence, R. I. 02912. For its support, an operational fund is being set up to which industrial organizations may contribute. To date, contributions of the following companies are gratefully acknowledged:

AVCO RESEARCH LABORATORY; EVERETT, MASS.,
BELL TELEPHONE LABORATORIES, INC.; NEW YORK, N. Y.,
THE BRISTOL COMPANY; WATERBURY, CONN.,
CALIFORNIA RESEARCH CORPORATION; SAN FRANCISCO, CAL.,
CURTISS WRIGHT CORPORATION; AIRPLANE DIVISION; BUFFALO, N. Y.,
EASTMAN KODAK COMPANY; ROCHESTER, N. Y.,
GENERAL ELECTRIC COMPANY; SCHENECTADY, N. Y.,
GULF RESEARCH AND DEVELOPMENT COMPANY; PITTSBURGH, PA.,
HUGHES AIRCRAFT COMPANY; CULVER CITY, CAL.,
INTERNATIONAL BUSINESS MACHINES CORPORATION; NEW YORK, N. Y.,
LEEDS & NORTHRUP COMPANY; PHILADELPHIA, PA.,
LOCKHEED AIRCRAFT CORPORATION; BURBANK, CAL.,
PRATT & WHITNEY DIVISION, NILES-BEMENT-POND COMPANY; WEST HARTFORD, CONN.,
THE RAND CORPORATION; SANTA MONICA, CAL.,
REPUBLIC AVIATION CORPORATION; FARMINGDALE, LONG ISLAND, N. Y.,
TRW INC.; REDONDO BEACH, CAL.,
UNITED AIRCRAFT CORPORATION; EAST HARTFORD, CONN.,
WESTINGHOUSE ELECTRIC AND MANUFACTURING COMPANY; PITTSBURGH, PA.

The QUARTERLY prints original papers in applied mathematics which have an intimate connection with application in industry or practical science. It is expected that each paper will be of a high scientific standard; that the presentation will be of such character that the paper can be easily read by those to whom it would be of interest; and that the mathematical argument, judged by the standard of the field of application, will be of an advanced character.

Manuscripts (one copy only) submitted for publication in the QUARTERLY OF APPLIED MATHEMATICS should be sent to the Editorial Office, Box 1885, Brown University, Providence, R. I. 02912, either directly or through any one of the Editors or Collaborators. In accordance with their general policy, the Editors welcome particularly contributions which will be of interest both to mathematicians and to engineers. Authors will receive galley proofs only. The authors' institution will be requested to pay a publication charge of $10.00 per page which, if honored, entitles them to 100 free reprints. Instructions will be sent with galley proofs.

The subscription price for the QUARTERLY is $15.00 per volume (April-January). Single issues can be purchased at $6.00, and back volumes at $15.00 per volume as far as they are available. Subscriptions and orders for single issues must be addressed to: American Mathematical Society, P. O. Box 6248, Providence, R. I., 02904.

Second-class postage paid at Providence, Rhode Island, and at Richmond, Virginia

---

WILLIAM BYRD PRESS, INC., RICHMOND, VIRGINIA

Reprinted 1976 in U.S.A.

SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ($'$), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\sqrt{\ }$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp\,[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2 + b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos(\pi x/2b)}{\cos(\pi a/2b)} \text{ is preferable to } \frac{\cos\dfrac{\pi x}{2b}}{\cos\dfrac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u\, du \text{ is preferable to } \int \frac{\sin u}{u}\, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t(a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (*for books*—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354–372(1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten*.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.

CONTENTS

—BOOK REVIEW SECTION—

Analytical treatment of one-dimensional Markov processes. By Petr Mandl. Springer-Verlag New York, Inc., New York, 1968. xx + 192 pp. $9.00.

This book is a presentation of the theory of one-dimensional diffusion processes (such as one-dimensional Brownian motion) leading up to the author's fairly complete results on the optimal stochastic control of these processes. The author considers Markov processes on a finite of infinite interval $I = [r_0, r_1]$ with an infinitesimal generator of Feller type $D_m D_p^+$, which includes in particular processes with generators of the form $a(x)d^2/dx^2 + b(x)d/dx$. These processes diffuse in the interior of I somewhat like Brownian motion, and on attaining a boundary point of I are either (i) absorbed or obliterated (the process "dies") or (ii) returned to the interior of I via a reflection or a discrete jump.

The book is divided into six chapters. The first three chapters contain a well-motivated discussion of Markov processes in general and one-dimensional diffusion processes in particular, assuming only a basic familiarity with probability and measure theory. The next two chapters contain various asymptotic results. Included are theorems about stationary measures (tendency to a steady state for a one-dimensional diffusion), asymptotic theorems about absorption probabilities and time spent in a subinterval, and various ergodic theorems for the process relative to a stationary measure. Most of the results here and in the final chapter are in the literature (due mostly to the author), but many of the results are new.

The author's results on optimal control theory appear in the last chapter. Here the generator of the process is of the form $a(x,z)d^2/dx^2 + b(x,z)d/dx$; a "control" is a piecewise continuous function $z = z(x)$ by which we control the diffusion and drift rates of the process. A cost is assumed which is the sum of a continuous cost for diffusing in the interior (of the form $\int_0^t c(x_s,z),ds$ where $\{x_s\}$ is the process) plus the sum of discrete costs resulting from jumps from the endpoints of I into the interior, or of "dying". An optimal control is one which minimizes the expected total cost in the non-conservative case (i.e. the process eventually dies or is absorbed) or the almost-sure average cost per unit time in the conservative (or ergodic) case. Criteria for optimal controls which are essentially formulas are derived in both cases.

The author's English is awkward at times, but there seem to be few typographical errors. Examples and problems, which are excellent for illustrating the material, are found at the end of each chapter.

STANLEY SAWYER (*Providence, R. I.*)

Elements of optimal control. By S. J. Citron. Holt, Rinehart, and Winston, New York, 1969. xiii + 266 pp. $10.95.

This is an introductory textbook on deterministic optimal control written for a one-semester course at the senior/first-year-graduate-student level. Some knowledge of ordinary differential equations and linear algebra is assumed, but no previous knowledge of control systems is required.

Ch. 1 (Introduction) discusses closed-loop vs. open-loop control ("control law" vs. "control function"), and performance indices.

Ch. 2 (Formative concepts) first treats *minima of functions* with inequality and/or equality constraints and introduces Lagrange multipliers. Then it treats the *calculus of variations* using classical nomenclature and the Lagrange formulation (minimizing definite integrals) ending with the case where there are differential constraints among the state variables. This reviewer felt the pedagogy was weak here since rather advanced topics (such as conjugate points, corner conditions, and the second variation) were introduced immediately; this despite the author's statement (p. 39), "It is believed that the needs of the engineering reader are often best served by first learning to use basic tools before complicating the subject to the extent that interest and perhaps utility are lost." In fact the first example (p. 38) is misleading since the stationary point is a local maximum instead of a local minimum; the path of the second example contains a conjugate point. Even though both of these examples are later explained in the chapter, it seems to this reviewer a difficult way to get the student into the subject.

Ch. 3 (Development through the control formulation). The first seven of nine sections still uses classical calculus of variations nomenclature. Again advanced topics are introduced immediately: variable end time problems, strong variations, the Weierstrass condition, and the Legendre-Clebsch condition. The last two sections introduce the modern control formulation with $\dot{x} = f(x, u, t)$; the equivalence of the Weierstrass condition to the Pontryagin Maximum Principle for problems without path constraints is demonstrated very nicely. A fourth order example, the lunar ascent problem, is treated thoroughly and clearly.

Ch. 4 (The extension to control variable inequality constraints). The first three of the eight sections of this chapter seem to belong in Ch. 3; numerical solution of the two-point boundary value problem is discussed and carried out for a chemical reactor problem with three state variables, by iterating on the unknown initial values to satisfy given final values. In later sections the problem is repeated with bounds on the control.

Ch. 5 (State variable inequality constraints). The "road-building" example here seems almost too simple to be instructive; on the other hand the minimum-weight variable-width beam problem is quite challenging and interesting since the location of the point of maximum deflection (which is constrained) is unknown, *a priori*.

Ch. 6 (Direct methods—the technique of steepest descent). The first half of this chapter deals with parameter optimization by steepest descent leading up to the corresponding technique in the calculus of variations. The lunar ascent problem, introduced in Ch. 3, is treated thoroughly as an example.

Ch. 7 (Dynamic programming optimal control). Discrete-step and continuous-time dynamic programming are introduced with simple examples and the relation to calculus of variations is discussed. The reviewer would quibble with statement on p. 234 regarding relative ease of computational solutions using dynamic programming and calculus of variations with inequality constraints.

Ch. 8 (Optimal feedback control) treats the problem of finding the control law for minimizing a quadratic performance index with linear differential equation constraints and linear terminal constraints. The use of this theory for perturbation control of nonlinear systems ends the book.

The book fulfills the author's intent of introducing the subject in a clear manner. This is done by use of several well-chosen examples mixed with straightforward derivations. However, the reviewer was sorry to see that there were no problems for the reader to solve. Also, singular problems were given only one page (p. 150).

A. E. Bryson (*Stanford*)