SEPARATION-OF-VARIABLES SOLUTION FROM THE SCHWARZ-CHRISTOFFEL TRANSFORMATION*

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I. Introduction. Two of the best-known methods in two-dimensional potential theory are based respectively on the Schwarz-Christoffel conformal transformation when the domain boundary is a polygon and on the separation of variables in polar coordinates when the boundary is a wedge or a sector of a circle. These complex variable and real variable methods are of sufficient practical importance that they are both discussed in numerous standard science or engineering texts, e.g. [1]–[3], without, however, an indication of a relation between the two methods. Actually, the separation-of-variables method must be applicable to polygonal domains because, as shown in Fig. 1 by the dotted arcs around the points m and n, any point on the boundary of a polygon can be regarded as the vertex of a sector of a circle, almost every boundary point corresponding to the vertex of a semicircle. The first motivation for writing this article is to exhibit the relation between the two methods by deriving the real separation-of-variables expansion from the complex Schwarz-Christoffel transformation. This derivation is also of practical value for it yields the coefficients of the separation-of-variables solution in terms of the more easily evaluated Schwarz-Christoffel parameters, the series representation of the potential in terms of trigonometric functions often being more useful than the implicit Schwarz-Christoffel representation.

The second motivation arises from the fact that published applications of the Schwarz-Christoffel transformation are customarily based on those relatively few cases where the basic differential form of that transformation is integrable in terms of standard functions. This implied restriction to integrable forms is reinforced by textbook summaries which state that the Schwarz-Christoffel transformation is useful provided that the differential form is integrable in terms of elementary functions [4] or that analytic methods fail when an integration in terms of standard functions is not possible [5]. It is therefore worthwhile by way of examples to show a class of analytic solutions that can be generated when such an integration is impossible and also to show that obtaining the integral is not always a desirable approach even when possible.

II. The method. In Fig. 1 the field at the corner may cause electrical breakdown if too large a potential is applied to the linear anode. (Alternatively Fig. 1 represents current crowding [12] or heat flow in a planar integrated circuit.) The Schwarz-Christoffel conformal transformation which maps the upper half of the w plane onto the interior of the polygon can be taken without loss of generality as [1]–[3], [5], [6]

$$c\, dz(w)/dw = w^{\alpha}(w - 1)^{-1}, \quad 0 \leq \alpha \leq 1. \tag{1}$$

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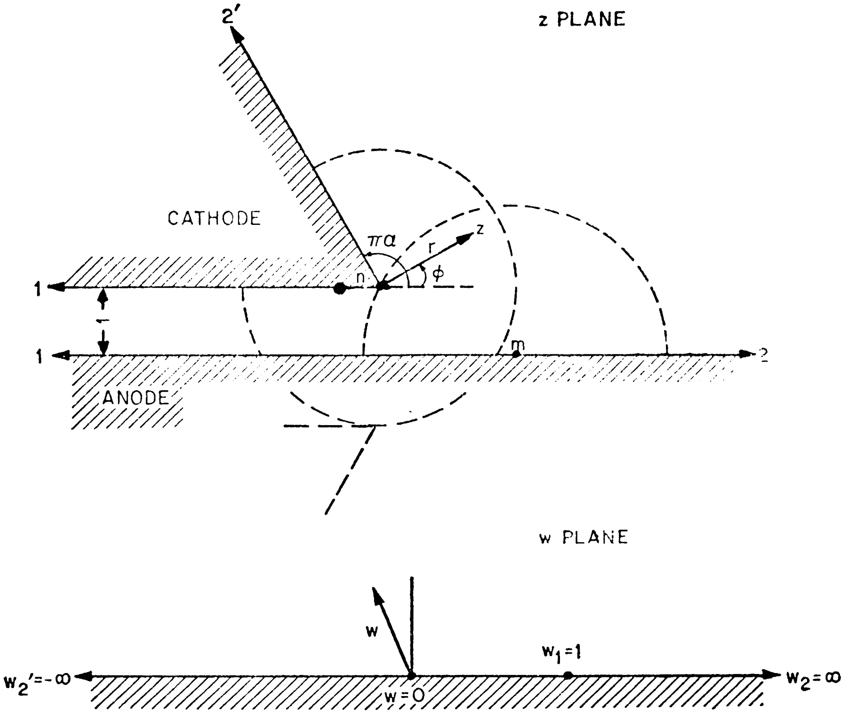

FIG. 1. A sharp-edged cathode

The z- and w-plane origins are taken as the corner and its image respectively, $\pi\alpha$ is the angle turned through at the corner when traversing the polygon boundary in a clockwise direction, and c is a constant.

When α is a rational number, m/n, the indefinite integral of Eq. (1) is

$$\begin{aligned} cz = {} & \frac{n}{m}\, w^{m/n} + \ln\,(w^{1/n} - 1) + (1 - s)(-1)^m \ln\,(w^{1/n} + 1) \\ & + \sum_{\nu=1,\cdots,t-1} \Big[\cos\,(2\pi\nu m/n) \ln\,(w^{2/n} - 2w^{1/n} \cos\,\{2\pi\nu/n\} + 1) \\ & - 2 \sin\,(2\pi\nu m/n) \operatorname{arctg} \left(\frac{w^{1/n} - \cos\,(2\pi\nu/n)}{\sin\,(2\pi\nu/n)} \right) \Big] \end{aligned} \tag{2}$$

where $n = 2t + s$, $s = 0$ or 1. This expression is not particularly convenient to manipulate. Also, the physical dependence on the angle α is obscured by the fact that n and m are separately scattered throughout the sum; e.g., the number of terms is proportional to n. As n and m are adjusted so that their ratio comes ever closer to some arbitrary irrational value of α, the number of terms in Eq. (2) increases without limit. Finally, there is the usual difficulty that w rather than z is the independent variable.

Rejecting the indefinite integral approach, we start again with Eq. (1). The constant c equals π as is easily found by integrating around a semicircle of vanishingly small radius centered at $w = 1$. If $(1 - w)^{-1}$ is expanded in a power series at $w = 0$, Eq. (1) becomes

$$\pi e^{i\pi} \frac{dz(w)}{dw} = \sum_{n=0}^{\infty} w^{\alpha+n} \tag{3}$$

TABLE I

Some Schwarz-Christoffel inversion coefficients

If

$$c\,\frac{dz(w)}{dw} = w^{\alpha} \prod_{m=1}^{p} (w - w_m)^{\alpha_m}, \qquad \alpha, \alpha_m \lesseqgtr 0, \qquad w_m \neq 0, \qquad \operatorname{Im} w > 0, \qquad z(0) = 0$$

Then

$$w(z) = \sum_{n=1}^{\infty} D_n(\alpha, \alpha_1, w_1, \cdots, \alpha_p, w_p)[(1 + \alpha)\lambda c z]^{n/(1+\alpha)}, \quad \text{neighborhood} \quad z = 0$$

where

$$\lambda = \prod_{m=1}^{p} (\gamma_m \, |w_m|)^{-\alpha_m}; \qquad \gamma_m = e^{i\pi}, \quad w_m > 0; \qquad \gamma_m = 1, \quad w_m < 0$$

and, for $p \leq 2$,

$$D_1 = 1, \qquad D_2 = \frac{1}{2 + \alpha}\left(\frac{\alpha_1}{w_1} + \frac{\alpha_2}{w_2}\right)$$

$$D_3 = \frac{4 + \alpha}{2(2 + \alpha)^2}\left(\frac{\alpha_1}{w_1} + \frac{\alpha_2}{w_2}\right)^2 - \frac{1}{3 + \alpha}\left(\frac{\alpha_1(\alpha_1 - 1)}{2w_1^2} + \frac{\alpha_2(\alpha_2 - 1)}{2w_2^2} + \frac{\alpha_1\alpha_2}{w_1 w_2}\right)$$

$$D_4 = \frac{(3 + \alpha)(5 + \alpha)}{3(2 + \alpha)^3}\left(\frac{\alpha_1}{w_1} + \frac{\alpha_2}{w_2}\right)^3 - \frac{5 + \alpha}{(2 + \alpha)(3 + \alpha)}\left(\frac{\alpha_1}{w_1} + \frac{\alpha_2}{w_2}\right)\left(\frac{\alpha_1(\alpha_1 - 1)}{2w_1^2} + \frac{\alpha_2(\alpha_2 - 1)}{2w_2^2} + \frac{\alpha_1\alpha_2}{w_1 w_2}\right)$$
$$+ \frac{1}{4 + \alpha}\left(\frac{\alpha_1(\alpha_1 - 1)(\alpha_1 - 2)}{6w_1^3} + \frac{\alpha_2(\alpha_2 - 1)(\alpha_2 - 2)}{6w_2^3} + \frac{\alpha_1(\alpha_1 - 1)}{2w_1^2}\frac{\alpha_2}{w_2} + \frac{\alpha_2(\alpha_2 - 1)}{2w_2^2}\frac{\alpha_1}{w_1}\right)$$

and, after integrating term by term,

$$\pi e^{i\pi} z(w) = \frac{w^{\alpha+1}}{\alpha+1} \sum_{n=0}^{\infty} \frac{\alpha+1}{\alpha+n+1} w^n. \tag{4}$$

The function $u(w)$,

$$u(w) \equiv [\pi(\alpha+1)e^{i\pi}z(w)]^{1/(1+\alpha)} = w\left(\sum_{n=0}^{\infty} \frac{\alpha+1}{\alpha+n+1} w^n\right)^{1/(1+\alpha)} = w + \sum_{n=2}^{\infty} d_n(\alpha)w^n, \tag{5}$$

is an analytic function of w with $du/dw \neq 0$ at $w = 0$. Thus by a well-known theorem of complex variables w is an analytic function of u at $w = 0$ and can therefore be expanded in a power series at that point:

$$w(z) = \sum_{n=1}^{\infty} D_n(\alpha)u^n = \sum_{n=1}^{\infty} D_n(\alpha)[\pi(\alpha+1)e^{i\pi}z]^{n/(1+\alpha)} \tag{6}$$

where the D's are real rational functions of α, e.g.

$$D_1(\alpha) = 1, \qquad D_2(\alpha) = -(2+\alpha)^{-1}, \qquad D_3 = \text{etc.} \tag{7}$$

The inversion of $z(w)$ is straightforwardly generalized to polygons of any number of sides and the first four D's are given in Table I for polygons of not more than four sides. The present example is the case $\alpha_1 = -1$ and $\alpha_2 = 0$, while expansion from the point m in Fig. 1 can be obtained using the substitutions $\alpha = 0$, $\alpha_1 = -1$, and $\alpha_2 \to \alpha$ in Table I.

The region of convergence of Eq. (6) or of the analogous expansion of $w(u(z))$ for any other polygon can be determined by inspection of the polygon, as the following argument shows. $w(u)$ is an analytic function of u at $u = z = 0$. Thus the region of convergence of the power series in u is bounded by a circle passing through that singularity of $w(u)$ which is nearest to $u = 0$. But $u = (\text{const.})z^{\text{const.}}$. Thus (a) the series in z also converges inside a circle (sector of a circle inside the polygon) and (b) except for the point $u = z = 0$ the singularities of $w(z)$ are mapped onto the singularities of $w(u)$ without changing their relative distances from the point $u = z = 0$. Now the singularities of $w(z)$ are simply the corners of the polygons and, by the principle of symmetry [7], the reflections and repeated reflections of those corners across polygon faces and across reflections of the polygon faces. It thus follows that the radius of convergence of the series expansion of $w(z)$ is the distance to the nearer of the following two points:

A. The nearest corner accessible by a straight line which does not go outside the polygon. An example is the dotted arc through the corner when the center of expansion is point m in Fig. 1. This condition always takes precedence when the center of expansion is a noncorner point on the polygon boundary.

B. The reflection of the center of expansion across the nearest (perpendicular distance) of certain polygon faces. The certain faces are any for which the perpendicular distance line has nonzero length, does not go outside of the polygon, and intersects the polygon face (rather than the extension of the face.) This condition is applicable only when the corner angle at the center of expansion is nonzero ($\alpha \neq 0$ in Table I). The corner is an example in Fig. 1 when $0 < \alpha \leq 1$. Thus Eq. (6) converges in the polygon interior where $|z| < 2$.

For convenience we will speak of the convergence-determining singularity as the "nearest" singularity although irrelevant singularities on other sheets may actually be nearer in the sense that their $|z|$ coordinate is less.

Let the real part of $\mathcal{V}(w)$ be the physical potential V which is taken as unity on the linear anode and zero on the cornered cathode. Physically, all sources of the field are on the anode or cathode and thus we require that $d\mathcal{V}/dw \to 0$ as $|w| \to \infty$. Then from $\nabla^2\mathcal{V}(w) = 0$, Im $w > 0$, and Eq. (6)

$$\begin{aligned} \mathcal{V}(z) &= i\pi^{-1} \ln [1 - w(z)] = -i\pi^{-1} \sum_{n=1}^{\infty} n^{-1}w^n(z) \\ &= -i\pi^{-1} \sum_{n=1}^{\infty} P_n(\alpha)[\pi(\alpha + 1)e^{i\pi}z]^{n/(1+\alpha)}, \qquad |z| < 2 \end{aligned} \tag{8}$$

where $P_1 = 1$, $P_2 =$ etc. In this example the region of convergence of $\mathcal{V}(z) = \mathcal{V}(w(z))$ is determined by the nearest singularity of $w(z)$ and hence is the same as that of Eq. (6). In general the nearest singularity of $\mathcal{V}(w)$ will be determining if nearer. If the nearest singularities of $w(z)$ and $\mathcal{V}(w)$ are at the same w point, they may cancel and yield a region of expansion of $\mathcal{V}(z)$ extending past a polygon corner.

The complex field $\mathcal{E}$ is the complex conjugate of the derivative of $\mathcal{V}(z)$, and thus from Eq. (8) the (normal) field on the anode face is

$$|\mathcal{E}(r, \alpha)| = \sum_{n=1}^{\infty} (\pm 1)^{n+1}Q_n(\alpha)[(1 + \alpha)\pi r]^{(n-\alpha-1)/(1+\alpha)}, \qquad 0 \le \alpha \le 1, \quad 0 < r < 2 \tag{9}$$

where

$$Q_1 = 1, \qquad Q_2 = \frac{\alpha}{2 + \alpha}, \qquad Q_3 = \frac{\alpha(2\alpha^2 + 5\alpha - 1)}{2(2 + \alpha)^2(3 + \alpha)}, \qquad Q_4 = \text{etc.} \tag{10}$$

Here $z = re^{i\varphi}$ and r is measured along the outer (upper sign) or inner (lower sign) cathode face. $|\mathcal{E}(r, 0)| = 1$ everywhere inside the parallel-plate capacitor as given by convergence condition A.

It can be shown that when $|\mathcal{E}| > 2$ the leading term of Eq. (9) is already within about 10% of the exact result, and thus one term may be sufficient near the corner if it is sharp. The first term of the real part of Eq. (8) is, with $z = re^{i\varphi}$,

$$\text{Re } \mathcal{V}(z) \equiv V(r, \varphi) = \pi^{-1}[(1 + \alpha)\pi r]^{1/(1+\alpha)} \sin ((\varphi + \pi)/(1 + \alpha)). \tag{11}$$

For $V =$ const., Eq. (11) becomes an expression for $r(\varphi)$, an equipotential contour which we take as a new effective cathode with a rounded corner following one of the standard methods for treating rounded corners [2], [6]. Some manipulation reveals that the magnitude of the maximum field on a corner of this type is

$$(\alpha\pi R)^{-\alpha/(1+\alpha)} \tag{12}$$

where R is the minimum radius of curvature of the rounded effective corner. This completes the first example.

Even when the indefinite integral of the differential Schwarz-Christoffel transformation cannot be expressed in terms of standard functions, it is still sometimes possible to relate c and the w_i's to the polygon dimensions using standard functions. (For $\alpha \neq 0$ these relations depend only on *definite* integrals between singularities.) For a second example consider Fig. 2 which reduces to Fig. 1 when $\alpha_1 = 0$. Let $w = 0$ be the corner

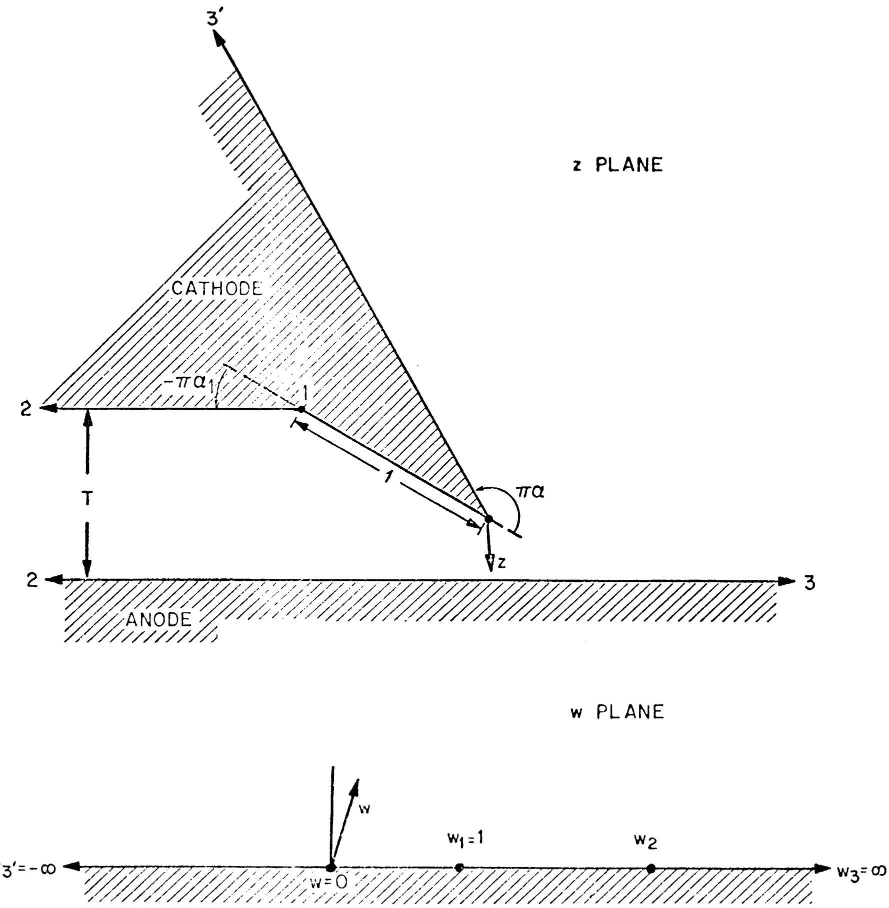

FIG. 2. Detail of an electron device

of angle α. Then if we set $w_1 = 1$ and $w_3 = \pm\infty$, the Schwarz-Christoffel transformation becomes [1]–[3], [5], [6]

$$c\, dz(w)/dw = w^{\alpha}(w - 1)^{\alpha_1}(w - w_2)^{-1}. \tag{13}$$

Although the indefinite integral of Eq. (13) is not apparently expressible in terms of standard functions, the definite integral from $w = 0$ to $w = 1$, corresponding to $z = 0$ to $z = \exp(-i\pi(1 - \alpha_1))$, yields [10]

$$c = \frac{\Gamma(\alpha + 1)\Gamma(\alpha_1 + 1)}{w_2\Gamma(\alpha + \alpha_1 + 2)} F(1, \alpha + 1; \alpha + \alpha_1 + 2; 1/w_2) \tag{14}$$

where Γ and F are the gamma and hypergeometric functions respectively. In analogy to the first example, integration around a semicircle at $w = w_2$ results in

$$T = (\pi/c)w_2^{\alpha}(w_2 - 1)^{\alpha_1} \tag{15}$$

$$= \pi \frac{w_2^{\alpha+1}(w_2 - 1)^{\alpha_1}\Gamma(\alpha + \alpha_1 + 2)}{\Gamma(\alpha + 1)\Gamma(\alpha_1 + 1)F(1, \alpha + 1; \alpha + \alpha_1 + 2; 1/w_2)} \tag{16}$$

where Eq. (14) was substituted into Eq. (15) to yield Eq. (16). For a given T, α, and α_1, Eq. (16) can be used to find w_2. Then c can be obtained from Eq. (15) or (14). Except for the implicit form of Eqs. (15) and (16), the method proceeds as in the first example using, for example, Table I. With the usual physical assumption that the polygon does not overlap itself, the expansion of $w(z, \alpha, \alpha_1)$ as a series in z centered at the corner of angle α converges in the polygon interior with α_1 in the range $-1 < \alpha_1 \leq 1$ when $|z|$ is less than the least of any of the following limits that are applicable to the particular values of α and α_1,

(1) infinity when $\alpha_1 = \alpha = 0$ (condition A),
(2) unity when $\alpha_1 \neq 0$ (condition A),
(3) $2(T + \sin \pi\alpha_1)$ when $-1 < \alpha_1 < 1/6$ and $\alpha \neq 0$ (condition B across face 2-3), or
(4) $2 \sin (\pi + \pi\alpha_1)$ when $-1 < \alpha_1 \leq -5/6$ (condition B across face 1-2).

Finally, there are polygons for which c and the w_i's cannot be related to the polygon dimensions using standard functions. Numerical methods are then required to obtain these relations and several exist in the literature [5], [8], [9]. In particular [8] contains an extended discussion. We emphasize that even if these parameters are related numerically, the resulting series in z is an exact analytic representation of the solution for the figure to which it applies and this figure is in turn as close to the original polygon as one cares to achieve numerically. Incidentally the whole procedure involves only standard computer library functions and thus one can write a program which yields as an output an analytic solution for a fairly arbitrary polygon. Also, analytical solutions near corners or other boundary singularities are valuable supplements to and substitute boundary conditions for numerical methods which often yield satisfactory solutions everywhere except near singularities in the field.

Equivalence with the separation-of-variables representation can be demonstrated as follows: $w(z)$ and functions of $w(z)$ analytic at $w = 0$ have been expanded in powers of $z^{1/(1+\alpha)}$. If $z = re^{i\theta}$, the real and imaginary parts of $z^{n/(1+\alpha)}$ are

$$r^{n/(1+\alpha)} \cos \frac{n\theta}{1+\alpha} \quad \text{and} \quad r^{n/(1+\alpha)} \sin \frac{n\theta}{1+\alpha} \tag{17}$$

which are product solutions of Laplace's equation separated in polar coordinates. Eq. (11) is, for example, the leading term of

$$V(r, \varphi) = \sum_{n=1}^{\infty} A_n(\alpha) r^{n/(1+\alpha)} \sin \frac{n\theta}{1+\alpha}, \qquad \theta = \varphi + \pi. \tag{18}$$

Finally, we note that when $\alpha = 0$ the expansion in z reduces to a power series in z.

III. Extensions. So far it has been assumed that $\mathcal{V}(w)$ is analytic at $w = 0$. If the real potential V on the boundary has a finite discontinuity at $w = 0$, then $\mathcal{V}(w)$ may be analytic at $w = 0$ except for a term proportional to $\ln w$. Now $\mathcal{V}(w) \sim \ln w$ implies $\mathcal{E}^*(z) \sim (w(z))^{-1}\, dw(z)/dz$ which can be again expanded as a series in z.

When the boundary consists of any finite number of constant-potential sections, $\mathcal{V}(w)$ can be expressed explicitly in terms of simple functions [2], [5], [6], a case of great [illegible] importance. When $\mathcal{V}(w)$ cannot be expressed in terms of useful functions or when an inhomogeneous term is present (Poisson's equation), the Green's function method in the half plane is a sometimes useful alternative to mapping onto the unit circle [6]. For example, with Dirichlet boundary conditions $G(w, w')$, the complex potential at w due to a point charge at w', is easily found in the half plane using the image-charge method. Then $\mathcal{G}(z, z') \equiv G(w(z), w'(z'))$ is the Green's function in the z plane [11] which can be expanded as a single or double series in z and/or z' using Table I or its extension. Here z and z' are measured from their individually chosen centers of expansion. Similarly the real Green's function can be expanded in the form of Eq. (17). Although more than one expansion from boundary or interior points is necessary to cover the entire interior of most polygons, in practice a single expansion is often sufficient to cover that part of the boundary where the prescribed potential is irregular or that part of the interior where the inhomogeneous term is nonzero.

Additional more or less similar extensions can be constructed by considering the exterior polygon problem, other boundary conditions, analytic continuation of the solution, and mapping functions other than the Schwarz-Christoffel transformation [13].

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