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A WEAK MOMENTUM SOURCE IN A UNIFORM STREAM*

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Abstract. The interaction of a momentum source with a viscous uniform stream is studied. Taken separately, each flow is described by an exact solution of the full Navier-Stokes equations. Their combined effect can be described in terms of two non-dimensional parameters related to the strength of the source and the distance from it. A perturbation solution for the case of a weak source is attempted and the initial terms are found. It is shown that the results are closely related to the Oseen expansion for viscous streaming past a spheroid and that when the momentum of the source opposes that of the stream a closed streamline of elliptical shape is formed.

1. Introduction. The leading terms in the Oseen expansions for the sphere (Proudman and Pearson [4]), and the spheroid (Breach [1]), are very similar in form. This is true even when the spheroid is very slender and suggests that the mere presence of a body, rather than its particular shape, is the essential factor affecting the flow in the large. From this overall point of view, when a body is present in a uniform stream it acts as a sink of momentum. The details of how this momentum is destroyed are built into the boundary conditions on the surface of the body and are of local interest only. Therefore a fundamental problem for investigation is that of a momentum source, or sink, placed in a uniform stream. This is a simpler problem than that of streaming past a finite body in that there are no fixed boundaries present and analytical difficulties in satisfying conditions thereon do not arise.

The momentum source in the absence of the uniform stream is described by a non-trivial exact solution of the full Navier-Stokes equations. The uniform stream alone corresponds to another exact solution. Hence when the two interact there is the possibility of constructing the solution by perturbations from exact solutions of the full non-linear equations. This would seem to be more satisfactory than a perturbation starting from exact solutions of linearised equations.

The problem depends on four parameters: the source strength, the viscosity, the velocity of the uniform stream, and the distance from the origin. These can be combined to give just two fundamental parameters and all the conditions that the solution must satisfy are expressible in terms of limiting values of these two.

2. A solution of the Navier-Stokes equations corresponding to a momentum source. By requiring the momentum flux across any sphere around the origin to be constant and imposing axial symmetry, Landau [3] constructed an exact solution of the Navier-Stokes equations (see Landau and Lifshitz [4]). This was subsequently discovered independently by Squire [7]. Sedov [5], by dimensional analysis, found this solution as one of a whole family of exact solutions in which the velocity components vary in-

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versely with the distance from the origin and the flux of momentum parallel to the axis of symmetry is constant. If ν is the viscosity and r, θ, ϕ, are spherical polar coordinates, the velocity components for this solution are given by

$$V_r = \frac{2\nu}{r}\left[\frac{A^2 - 1}{(A - \cos\theta)^2} - 1\right], \qquad V_\theta = -\frac{2\nu \sin\theta}{r(A - \cos\theta)}, \qquad V_\phi = 0 \tag{1}$$

where A is a constant such that $|A| > 1$.

If d is the density and Md the flux of momentum, parallel to the axis of symmetry, across a sphere about the origin, then

$$M = 16\pi\nu^2\left[A + \frac{4}{3}\cdot\frac{A}{(A^2 - 1)} - \frac{1}{2}A^2 \log\left(\frac{A + 1}{A - 1}\right)\right]. \tag{2}$$

This relates the constant A to the strength of the momentum source which will be measured by M. In this work M is small, which implies that A is large. Eq. (2) gives M explicitly in terms of A. It may be readily inverted to give A in terms of M. Thus

$$A^{-1} = \epsilon - \frac{17}{15}\epsilon^3 + \frac{466}{175}\epsilon^5 - \frac{2317}{525}\epsilon^7 + O(\epsilon^9), \qquad |A| > 1, \tag{3}$$

where $16\pi\nu^2\epsilon = M$.

3. Nondimensional parameters, equations of motion, and boundary conditions. It is now supposed that the momentum source is placed in a uniform stream of velocity U along the axis of symmetry so there will be an interaction between the two flows. The physical parameters involved are r, ν, U, M, and $\mu = \cos\theta$, of which the last is dimensionless. For the others

$$[r] = L; \qquad [\nu] = L^2T^{-1}; \qquad [U] = LT^{-1}; \qquad [M] = L^4T^{-2}. \tag{4}$$

There are just two basic nondimensional parameters which can be formed from these. They are

$$\lambda = M\nu^{-2} \quad \text{and} \quad \rho = Ur\nu^{-1} \tag{5}$$

corresponding to a nondimensional source strength and a nondimensional distance from the origin.

It is convenient to describe the motion in terms of ψ, the Stokes stream function, and l, the vorticity divided by the distance from the axis. In spherical polar coordinates the Navier-Stokes equations for the axisymmetric flow are

$$r^2\frac{\partial^2\psi}{\partial r^2} + (1 - \mu^2)\frac{\partial^2\psi}{\partial\mu^2} = -r^4(1 - \mu^2)l, \qquad \mu = \cos\theta; \tag{6}$$

and

$$r^2\frac{\partial^2 l}{\partial r^2} + 4r\frac{\partial l}{\partial r} + (1 - \mu^2)\frac{\partial^2 l}{\partial\mu^2} - 4\mu\frac{\partial l}{\partial\mu} = \frac{1}{\nu}\frac{\partial(\psi, l)}{\partial(r, \mu)}. \tag{7}$$

These are in physical variables. Equivalent nondimensional forms can be derived by using the nondimensional parameters λ and ρ.

The stream function, ψ, has dimensions L^3T^{-1}. Therefore $\psi(r\nu)^{-1}$ is dimensionless and must be a function of λ, ρ and μ. Likewise $[l] = L^{-1}T^{-1}$ so $lr^3\nu^{-1}$ is dimensionless and is also a function of λ, ρ and μ. From Eq. (2) λ is a function of A only and A can be used as a parameter in place of λ. Large values of A correspond to small values of λ. Thus

$$\psi(r\nu)^{-1} = F(A, \rho, \mu) \quad \text{and} \quad lr^3\nu^{-1} = G(A, \rho, \mu) \tag{8}$$

where F and G are functions to be determined.

It is assumed that M is small so the momentum source is a weak one. Keep ν fixed. Then as $M \to 0$, $A \to \infty$. If the source is switched off only the uniform stream remains, so

$$F(\infty, \rho, \mu) = \tfrac{1}{2}\rho(1 - \mu^2); \qquad G(\infty, \rho, \mu) = 0. \tag{9}$$

Again, if ν is kept fixed, then as $U \to 0$, $\rho \to 0$. This corresponds to the removal of the uniform stream to leave the momentum source only. Therefore

$$F(A, 0, \mu) = 2(1 - \mu^2)/(A - \mu); \qquad G(A, 0, \mu) = -4(1 - A^2)/(A - \mu)^3. \tag{10}$$

Eqs. (9) and (10) give conditions on F and G, which satisfy the equations

$$\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial F}{\partial \rho}\right) + (1 - \mu^2)\frac{\partial^2 F}{\partial \mu^2} = -(1 - \mu^2)G \tag{11}$$

and

$$\rho^4 \frac{\partial}{\partial \rho}\left(\rho^{-2} \frac{\partial G}{\partial \rho}\right) + (1 - \mu^2)\frac{\partial^2 G}{\partial \mu^2} - 4\mu \frac{\partial G}{\partial \mu} = \rho \frac{\partial(F, G)}{\partial(\rho, \mu)} + 3G \frac{\partial F}{\partial \mu} + F \frac{\partial G}{\partial \mu}. \tag{12}$$

These are homogeneous in the distance variable and A does not occur explicitly.

Since A is large a perturbation solution in terms of A^{-1} will be attempted.

5. The initial steps in the solution. It is assumed that

$$F = \sum_{n=0}^{\infty} A^{-n} F_n(\rho, \mu) \quad \text{and} \quad G = \sum_{n=0}^{\infty} A^{-n} G_n(\rho, \mu) \tag{13}$$

where the F_n and G_n are functions to be determined. By conditions (9),

$$F_0 = \rho(1 - \mu^2)2 \quad \text{and} \quad G_0 = 0. \tag{14}$$

The equations for F_1 and G_1 are found by substituting (13) into (4) and (12) and selecting the coefficient A^{-1}. This gives

$$\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial F_1}{\partial \rho}\right) + (1 - \mu^2)\frac{\partial^2 F_1}{\partial \mu^2} = -(1 - \mu^2)G_1 \tag{15}$$

and

$$\rho^4 \frac{\partial}{\partial \rho}\left(\frac{1}{\rho^2} \frac{\partial G_1}{\partial \rho}\right) + (1 - \mu^2)\frac{\partial^2 G_1}{\partial \mu^2} - 4\mu \frac{\partial G_1}{\partial \mu} = \rho \frac{\partial(F_0, G_1)}{\partial(\rho, \mu)} + 3G_1 \frac{\partial F_0}{\partial \mu} + F_0 \frac{\partial G_1}{\partial \mu}. \tag{16}$$

These are Oseen's equations in disguise. That for G_1 can be made separable by first extracting a factor exp $(\rho\mu/2)$ (Goldstein, [2]). Solutions for G_1 which decay as $\rho \to \infty$ are of the form

$$b_n \rho^{3/2} K_{n+1/2}(\rho/2) \frac{U_n(\mu)}{(1 - \mu^2)} \exp(\rho\mu/2), \qquad n \geq 1, \tag{17}$$

where $K_{n+1/2}(\rho)$ is a modified Bessel function of the second kind and $U_n(\mu)$ is the polynomial defined in terms of Legendre's polynomial, $P_n(\mu)$, by

$$U_n(\mu) = \int_1^{\mu} P_n(x)\, dx. \tag{18}$$

As $\rho \to 0$, $\rho^{3/2}K_{n+1/2}(\rho/2) = O(\rho^{1-n})$, so conditions (10) will be violated unless $b_n = 0$, $n \geq 2$. Therefore

$$G_1(\rho, \mu) = b_1\rho^{3/2}K_{3/2}(\rho/2)\frac{U_1(\mu)}{(1 - \mu^2)}\exp(\rho\mu/2). \tag{19}$$

The constant b_1 is determined by the conditions (10). The polynomials $U_n(\mu)$ are linearly independent. Therefore, since terms in A^{-1} are being considered, as $\rho \to 0$ the expansion of G_1 in terms of these polynomials must agree with the expansion of the coefficient of A^{-1} in $G(A, 0, \mu)$ when this is expanded in terms of the same polynomials. In general the expansion of $\lim_{\rho\to 0} G_n(\rho, \mu)$ in terms of $U_n(\mu)(1 - \mu^2)^{-1}$ has to match the expansion of the coefficient of A^{-n} in $G(A, 0, \mu)$.

Now, if $|\mu| \leq 1 < A$, then

$$\begin{aligned} G(A, 0, \mu) &= -4(1 - A^2)/(A - \mu)^3 \\ &= -2(1 - A^2)\frac{\partial^2}{\partial A\,\partial\mu}\frac{1}{(A - \mu)} \\ &= -2(1 - A^2)\frac{\partial^2}{\partial A\,\partial\mu}\left[\sum_{n=0}^{\infty}(2n + 1)P_n(\mu)Q_n(A)\right] \end{aligned} \tag{20}$$

by von Neumann's expansion for the Legendre functions of the first and second kind, $P_n(x)$ and $Q_n(x)$. But $U_n(\mu)$ satisfies

$$(1 - \mu^2)U_n''(\mu) + n(n + 1)U_n(\mu) = 0. \tag{21}$$

Therefore if $V_n(\mu) = \int_\mu^\infty Q_n(x)\,dx$, $|\mu| > 1$, is taken as a second solution of this equation, then

$$G(A, 0, \mu) = -2\sum_{n=1}^{\infty}(2n + 1)n^2(n + 1)^2V_n(A)\frac{U_n(\mu)}{(1 - \mu^2)}. \tag{22}$$

This expresses $G(A, 0, \mu)$ in the appropriate angle functions and at the same time expresses it in suitable functions of A. For large A, $V_n(A)$ has a series expansion in inverse powers of A such that $A^nV_n(A) = O(1)$.

Since $\lim_{\rho\to 0} G_1(\rho, \mu)$ is equal to the coefficient of A^{-1} in (22) and $U_1(\mu) = \frac{1}{2}(\mu^2 - 1)$,

$$V_1(A) = \frac{1}{2}\left[A - \frac{(A^2 - 1)}{2}\log\left(\frac{A + 1}{A - 1}\right)\right], \tag{23}$$

it follows that

$$b_1 = -4\pi^{-1/2}. \tag{24}$$

Therefore, if the closed expression for $K_{3/2}(\rho/2)$ is used,

$$G_1 = 2(\rho + 2)\exp[-\tfrac{1}{2}\rho(1 - \mu)]. \tag{25}$$

The equation for F_1 is then

$$\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial F_1}{\partial\rho}\right) + (1 - \mu^2)\frac{\partial^2 F_1}{\partial\mu^2} = -2(1 - \mu^2)(\rho + 2)\exp[-\tfrac{1}{2}\rho(1 - \mu)] \tag{26}$$

which has the particular integral

$$F_1 = 4\{1 - \exp[-\tfrac{1}{2}\rho(1 - \mu)]\}(1 + \mu)/\rho. \tag{27}$$

The coefficient of A^{-1} in $F(A, 0, \mu)$ is, by (10), $2(1 - \mu^2)$. From (27), $\lim_{\rho\to 0} F_1(\rho, \mu) = 2(1 - \mu^2)$, so F_1 as given above satisfies the correct condition for small ρ.

Thus the initial terms of the expansions for ψ and l are

$$\psi = r\nu\left\{\frac{\rho}{2}(1 - \mu^2) + \frac{4}{A\rho}(1 + \mu)\left[1 - \exp\left(-\frac{\rho}{2} + \frac{\rho\mu}{2}\right)\right] + O(A^{-2})\right\} \tag{28}$$

and

$$l = \frac{\nu}{r^3}\left\{\frac{2}{A}(\rho + 2)\exp\left(-\frac{\rho}{2} + \frac{\rho\mu}{2}\right) + O(A^{-2})\right\}. \tag{29}$$

If $A > 0$ these are the solutions for the case when the momentum source assists the stream. If $A < 0$ the momentum source opposes the stream.

6. The existence of a closed streamline. From (28), if the terms in $O(A^{-2})$ are neglected, the expansion of ψ for small ρ is

$$\psi = \frac{r\nu}{2A}[4 + \rho(A - 1 + \mu) + O(\rho^2)](1 - \mu^2). \tag{30}$$

For small ρ, $\psi = 0$ on the curve

$$Ur/\nu = \rho = -4/(A - 1 + \mu), \qquad \mu = \cos\theta. \tag{31}$$

For A large and positive (31) gives only negative values for ρ. Hence, when the source assists the stream there will be no zero streamline, other than the axis, near the origin. If, however, A is large and negative, then ψ will vanish on an ellipse with a focus at the origin. This occurs when the source opposes the stream.

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