CONTROLLABLE STATES IN THERMOELASTICITY*

By N. LAWS (*Lehigh University, Bethlehem, Pennsylvania*)

1. Introduction. The problem of determining all possible deformations which can be produced in every homogeneous, isotropic, incompressible elastic body by surface tractions alone was initiated, and largely solved, by Ericksen [1]. While Ericksen's problem lay dormant for more than a decade, there has, of late, been renewed interest in the problem but mainly in the context of elastic dielectrics [2] and thermoelasticity [3], [4], [5]. An important by-product of the work of Singh and Pipkin [2] was the discovery [6] of a further family of deformations of elastic bodies not included in Ericksen's [1] list. This family generalized a family of solutions found by Klingbeil and Shield [7].

Singh and Pipkin [6] designated the deformations sought by Ericksen as controllable deformations. Also, these authors described the resulting state of the elastic body as a controllable state. We use this terminology in our discussion of thermoelasticity.

In this note we wish to determine all controllable states of homogeneous, isotropic, incompressible thermoelastic bodies. Since controllable states are found by the inverse method it is important to know which variables are being prescribed along with the deformation. First, suppose that we look for controllable states with prescribed temperature. Then Petroski and Carlson [3] have shown that such states are necessarily at uniform temperature. Thus if we could solve Ericksen's problem for elastic bodies we could write down all controllable states with prescribed temperature. Second, suppose that we look for controllable states with prescribed heat flux. Then a statement by Singh and Pipkin [2, Section I.4] may give the impression that such states can be read off directly from that paper. The statement is incorrect. An examination of the constitutive equations used by these authors shows that the results in [2] apply immediately to thermoelasticity only in the circumstance that the response functions be independent of the temperature. In this note we obtain the complete list of controllable states with prescribed heat flux for incompressible isotropic thermo elastic bodies with temperature-dependent response functions. As might be expected, there are fewer controllable states for temperature-dependent response functions than there are for temperature-independent response functions.

2. Basic equations for thermoelasticity. Let $\boldsymbol{F}$ be the deformation gradient; then for an incompressible material, $\det \boldsymbol{F} = 1$. Let θ be the temperature, $\boldsymbol{h}$ be the heat flux and $\boldsymbol{T}$ be the Cauchy stress tensor. A usual set of constitutive equations for homogeneous, isotropic, incompressible thermoelastic bodies may be written in the form:[1]

$$\boldsymbol{T} = -p\boldsymbol{1} + \boldsymbol{S} = -p\boldsymbol{1} + \phi_1\boldsymbol{B} + \phi_2\boldsymbol{B}^2, \tag{2.1}$$

$$\operatorname{grad} \theta = (\psi_0\boldsymbol{1} + \psi_1\boldsymbol{B} + \psi_2\boldsymbol{B}^2)\boldsymbol{h} \tag{2.2}$$

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[1] See, for example, Truesdell and Noll [8].

where grad θ denotes the spatial temperature gradient and p is undetermined. Also

$$\phi_i = \phi_i(\theta, J_1, J_2), \qquad i = 1, 2, \tag{2.3}$$

$$\psi_i = \psi_i(\theta, J_1, J_2, J_3, J_4, J_5), \qquad i = 0, 1, 2, \tag{2.4}$$

where the invariants are given by

$$J_1 = \operatorname{tr} \boldsymbol{B}, \qquad J_2 = \operatorname{tr} \boldsymbol{B}^2, \qquad J_3 = \boldsymbol{h}\cdot\boldsymbol{h} \qquad J_4 = \boldsymbol{h}\cdot\boldsymbol{B}\boldsymbol{h}, \qquad J_5 = \boldsymbol{h}\cdot\boldsymbol{B}^2\boldsymbol{h}$$

with $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$. We note that (2.2) is the usual constitutive equation for the heat flux written in a different form. The usual practice[1] is to write $\boldsymbol{h}$ in terms of $\boldsymbol{B}$ and grad θ. Here, it is more convenient to express grad θ in terms of $\boldsymbol{B}$ and $\boldsymbol{h}$.

The equations of equilibrium and the energy equation are

$$\operatorname{div} \boldsymbol{S} = \operatorname{grad} p, \tag{2.5}$$

$$\operatorname{div} \boldsymbol{h} = 0. \tag{2.6}$$

3. The problem. We wish to determine all the exceptional states of homogeneous, isotropic incompressible thermoelastic bodies for which a prescribed deformation and a prescribed heat flux are such that the equations of equilibrium and the energy equation are satisfied for all functions ϕ, ψ. More precisely, we wish to determine all pairs $(\boldsymbol{B}, \boldsymbol{h})$, where $\boldsymbol{B}$ is a positive definite symmetric tensor and $\boldsymbol{h}$ a vector such that div $\boldsymbol{h} = 0$, for which there exist scalars p, θ such that

$$\operatorname{grad} p = \operatorname{div} (\phi_1\boldsymbol{B} + \phi_2\boldsymbol{B}^2), \tag{3.1}$$

$$\operatorname{grad} \theta = (\psi_0\boldsymbol{1} + \psi_1\boldsymbol{B} + \psi_2\boldsymbol{B}^2)\boldsymbol{h} \tag{3.2}$$

for all choices of ϕ, ψ given by (2.3) and (2.4). We say that such a pair $(\boldsymbol{B}, \boldsymbol{h})$ gives rise to a controllable state with prescribed heat flux.

Singh and Pipkin [2] have gone far toward solving this problem. In fact they have solved the problem completely for the special case when the response functions are independent of the temperature. However, not all the solutions given by these authors [2] are controllable when the response functions are allowed to depend upon the temperature. In this note we will obtain all controllable states with prescribed heat flux for temperature-dependent response functions.

We will solve the problem by listing the results of Singh and Pipkin [2] for temperature-independent response functions and showing how these results are modified when the response functions are allowed to depend upon the temperature. While the approach may not be the most elegant possible, it appears to have the attribute that it is the quickest way of solving the problem.

4. Controllable states with prescribed heat flux for temperature-independent response functions. In this section we use the results from [2] to write down a complete list of all controllable states[2] with prescribed heat flux when the response functions ϕ, ψ do not depend upon temperature. The analogous states in [2] are those with specified dielectric displacement field.

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[2] We also make use of the generalization of some of Ericksen's results mentioned by Fosdick [9].

We first note that the field equation used in [2] parallel our equations (2.5) and (2.6) and that the constitutive equations are (in our notation)

$$\mathbf{T} = -p\mathbf{1} + \boldsymbol{S}$$

$$\boldsymbol{S} = \phi_1 \boldsymbol{B} + \phi_2 \boldsymbol{B}^2 + \phi_3 \boldsymbol{h} \otimes \boldsymbol{h} + \phi_4(\boldsymbol{h} \otimes \boldsymbol{Bh} + \boldsymbol{Bh} \otimes \boldsymbol{h}) + \phi_5(\boldsymbol{h} \otimes \boldsymbol{B}^2\boldsymbol{h} + \boldsymbol{B}^2\boldsymbol{h} \otimes \boldsymbol{h}), \qquad (4.1)$$

together with (2.2), where each ϕ and ψ is a function of J_1, J_2, J_3, J_4, J_5. Next we observe that the conclusions of Singh and Pipkin [2, Part III] are wholly unaltered by the appearance of ϕ_3, ϕ_4, ϕ_5 and that ϕ_1, ϕ_2 can depend upon J_3, J_4, J_5. Hence, we may conclude that the following list of controllable states is complete when one adopts (2.1) instead of (4.1) as the constitutive equation for the stress, provided, of course, that the remaining ϕ's and ψ's are independent of θ.

We use capital letters for the coordinates in the natural state and small letters for the coordinates in the deformed configuration. The (nonzero) heat flux is specified by its physical components with respect to an appropriate orthonormal basis.

Family 0. Isochoric, homogeneous deformations with constant heat flux.

Family 1. Let (X, Y, Z) be cartesians and (r, θ, z) be cylindrical polars.

$$r^2 = 2AX + B, \qquad \theta = CY + DZ + G, \qquad z = EY + FZ + H,$$

$$A(CF - DE) = 1,$$

$$h_r = K/r, \qquad h_\theta = h_z = 0.$$

Family 2. Let (R, Θ, Z) be cylindrical polars and (x, y, z) be cartesians.

$$x = \tfrac{1}{2}AR^2 + B, \qquad y = C\Theta + DZ + G, \qquad z = E\Theta + FZ + H,$$

$$A(CF - DE) = 1,$$

$$h_x = K, \qquad h_y = h_z = 0.$$

Family 3. Let (R, Θ, Z), (r, θ, z) be cylindrical polars.

$$r^2 = AR^2 + B, \qquad \theta = C\Theta + DZ + G, \qquad z = E\Theta + FZ + H$$

$$A(CF - DE) = 1,$$

$$h_r = K/r, \qquad h_\theta = h_z = 0.$$

Family 4. Let (R, Θ, Φ), (r, θ, ϕ) be spherical polars.

$$r^3 = \pm R^3 + A, \qquad \theta = \pm\Theta, \qquad \phi = \Phi + C,$$

$$h_r = K/r^2, \qquad h_\theta = h_\phi = 0.$$

Family 5. Let (R, Θ, Z), (r, θ, z) be cylindrical polars.

$$r = AR, \qquad \theta = B \log R + C\Theta + D, \qquad z = EZ + F$$

$$A^2CE = 1,$$

$$h_r = h_\theta = 0, \qquad h_z = K.$$

In each case the temperature is determined by integrating (2.2).

5. Controllable states with prescribed heat flux for temperature-dependent response functions. Perhaps the easiest way of determining the controllable states with prescribed heat flux when the response functions depend upon the temperature is to start with the solutions of the preceding section. In particular, it is easily verified that Families 1, 2, 3 and 4 are again controllable.

The other two families require modification. We first consider Family 0. For a particular deformation of this type to be controllable, we see from (3.2) that $(\psi_0 \boldsymbol{1} + \psi_1 \boldsymbol{B} + \psi_2 \boldsymbol{B}^2)\boldsymbol{h}$ must be the gradient of a scalar, θ, for all ψ_0, ψ_1, ψ_2. To meet this requirement, it is necessary and sufficient that

$$\text{grad}\ \{(\psi_0 \boldsymbol{1} + \psi_1 \boldsymbol{B} + \psi_2 \boldsymbol{B}^2)\boldsymbol{h}\} \tag{5.1}$$

be symmetric for all ψ_0, ψ_1, ψ_2. If we remember that $\boldsymbol{B}$ and $\boldsymbol{h}$ are constant for Family 0, then (5.1) reduces to

$$\left(\frac{d\psi_0}{d\theta} \boldsymbol{1} + \frac{d\psi_1}{d\theta} \boldsymbol{B} + \frac{d\psi_2}{d\theta} \boldsymbol{B}^2\right)\boldsymbol{h} \otimes \text{grad}\ \theta.$$

With the help of (2.2), the above expression becomes

$$\left(\frac{d\psi_0}{d\theta} \boldsymbol{1} + \frac{d\psi_1}{d\theta} \boldsymbol{B} + \frac{d\psi_2}{d\theta} \boldsymbol{B}^2\right)\boldsymbol{h} \otimes (\psi_0 \boldsymbol{1} + \psi_1 \boldsymbol{B} + \psi_2 \boldsymbol{B}^2)\boldsymbol{h}. \tag{5.2}$$

For (5.2) to be symmetric for all ψ_0, ψ_1, ψ_2 it is apparent that, in particular, we need $\boldsymbol{Bh} \otimes \boldsymbol{h}$ to be symmetric. This in turn implies that $\boldsymbol{h}$ must be an eigenvector of $\boldsymbol{B}$. Furthermore, it is easy to see that if $\boldsymbol{h}$ is an eigenvector of $\boldsymbol{B}$, then (5.2) is always symmetric. Hence Family 0 must be modified in that the controllable states must have the heat flux an eigenvector of $\boldsymbol{B}$. With this restriction on $\boldsymbol{h}$ one readily verifies that the pressure p can be found to satisfy the equilibrium equations.

Finally we consider Family 5. We will show how Family 5 is modified simply by inserting this deformation into the field equations and finding under what circumstances it yields a solution. Substitution of the relevant data into (2.2) shows that the temperature is necessarily a function of z alone. Since all invariants are constant, this implies that ϕ_1, ϕ_2 become functions of z alone and that the equilibrium equations (2.5) become

$$\frac{\partial p}{\partial r} = \frac{A^2}{r}(1 - B^2 - C^2)[\phi_1 + \phi_2 A^2(1 + B^2 + C^2)],$$

$$\frac{\partial p}{\partial \theta} = 2\phi_1 BA^2 + 2\phi_2 BA^4(1 + B^2 + C^2),$$

$$\frac{\partial p}{\partial z} = \frac{\partial}{\partial z}(\phi_1 E^2 + \phi_2 E^4).$$

In order that there exist a scalar p satisfying these equations, it is necessary and sufficient that

$$\begin{aligned} &\frac{\partial}{\partial z}\left\{\frac{A^2}{r}(1 - B^2 - C^2)(\phi_1 + \phi_2 A^2)\right\} = 0, \\ &\frac{\partial}{\partial z}\{2\phi_1 BA^2 + 2\phi_2 BA^4(1 + B^2 + C^2)\} = 0, \end{aligned} \tag{5.3}$$

for all ϕ_1, ϕ_2. Since we also have $A^2CE = 1$, it is readily seen that (5.3) imply that

$B = 0$, $C = \pm 1$. This in turn implies that Family 5 reduces to a subfamily of the (modified) Family 0.

To summarize, we have the following list of controllable states:

Family 0. Isochoric, homogeneous deformations with the heat flux an eigenvector of the tensor $\boldsymbol{B}$.

Families 1, 2, 3, 4. As before.

In each case the temperature is determined by integrating (2.2).

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