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A NOTE ON ELASTIC-PLASTIC FLOW*

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Abstract. The plane plastic flow of an incompressible elastic perfectly-plastic solid that obeys the Mises yield condition and a properly invariant form of the Prandtl-Reuss equations is considered. It is shown that both the stress and velocity equations are hyperbolic and that the two families of characteristics are not coincident except for the limiting case of the rigid perfectly-plastic solid.

1. Introduction. We consider the quasi-static plane flow of a solid that has the following constitutive equations, referred to a fixed system of curvilinear coordinates x_i :

$$\frac{1}{2\mu}\frac{\mathfrak{D}s^i_j}{\mathfrak{D}t} = d^i_j - \frac{d^m_n s^n_m}{2k^2} s^i_j \quad \text{for} \quad J = s^m_n s^n_m = 2k^2, \tag{1}$$

$$\frac{1}{2\mu}\frac{\mathfrak{D}s^i_j}{\mathfrak{D}t} = d^i_j \quad \text{for} \quad J < 2k^2 \quad \text{or for} \quad J = 2k^2 \quad \text{and} \quad J^0 < 0; \tag{2}$$

and, since the solid is incompressible,

$$d^i_i = 0, \tag{3}$$

where μ and k are material constants, d^i_j , $s^i_j = \sigma^i_j - \frac{1}{3}\sigma^k_k\, \delta^i_j$ and σ^i_j are the components of the rate of deformation, stress deviator and stress tensors respectively and J^0 denotes the material derivative of J. The stress rate that appears in (1) and (2) is given by

$$\frac{\mathfrak{D}s^i_j}{\mathfrak{D}t} = \frac{\partial s^i_j}{\partial t} + s^i_{j,k}v^k + s^i_k w^k_{.j} - s^k_j w^i_{.k} , \tag{4}$$

where v^k are the components of velocity,

$$w^i_{.j} = \tfrac{1}{2}(u^i_{,j} - g^{ik}v_{j,k})$$

are the components of the spin tensor and subscript commas denote covariant differentiation. If μ and k are the shear modulus and shear yield stress, respectively, Eqs. (1) are a properly invariant form, suggested by Thomas [1], of the Prandtl-Reuss equations for a non-work-hardening solid.

The stress rate given by (4) is sometimes known as the corotational stress rate and is the stress rate following a material element and referred to a system of axes that rotates with an angular velocity equal to that of the element. Green [2] has noted that Eqs. (2) indicate a linear relationship between stress and logarithmic strain for simple extension. For infinitesimal strains integration of (2) yields the constitutive equations for a Hookean elastic solid. However, if μ/k is sufficiently small so that the strains may not be repre-

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sented by the infinitesimal strain tensor for all $J < 2k^2$, Eqs. (2) do not necessarily described elastic behavior but describe a form of hypo-elastic behavior [3]. Nevertheless we assume that Eqs. (1) are valid for elastic-plastic flow for $\mu/k > 1$.

For three-dimensional elastic-plastic flow only four of Eqs. (1) are independent and these four equations, Eq. (3), the yield condition

$$J = 2k^2 \tag{5}$$

and the three equilibrium equations provide nine equations for the determination of the nine unknowns σ_j^i, v^i.

Plane flow. In this section rectangular Cartesian coordinates (x, y, z) are used and the x, y and z components of velocity are denoted by u, v and w respectively. For plane flow independent of z and parallel to the (x, y) plane

$$\sigma_{xz} = \sigma_{yz} = w = d_{xz} = d_{yz} = d_{zz} = 0.$$

Since $d_{zz} = 0$ and the solid is incompressible, $s_{zz} = 0$ and $\sigma_{zz} = (\sigma_{xx} + \sigma_{yy})/2 = -p$ where $-p$ is the hydrostatic part of the stress tensor. Consequently $s_{xx} = -s_{yy}$ and the Mises yield condition (5) becomes

$$s_{xx}^2 + s_{xy}^2 = k^2. \tag{6}$$

There is only one independent Prandtl-Reuss equation for incompressible plane elastic-plastic flow, say

$$\frac{1}{2\mu}\frac{\mathfrak{D}s_{xx}}{\mathfrak{D}t} = d_{xx} - \frac{(d_{xx}s_{xx} + 2d_{xy}s_{xy} + d_{yy}s_{yy})}{2k^2}s_{xx},$$

which may be rewritten as

$$\frac{k^2}{\mu}\left(u\frac{\partial s_{xx}}{\partial x} + v\frac{\partial s_{xx}}{\partial y}\right) - 2(k^2 - s_{xx}^2)\frac{\partial u}{\partial x} - \left(\frac{k^2}{\mu} - s_{xx}\right)s_{xy}\frac{\partial u}{\partial y} + \left(\frac{k^2}{\mu} + s_{xx}\right)s_{xy}\frac{\partial v}{\partial x} = 0. \tag{7}$$

Eqs. (6) and (7), the equilibrium equations which may be put in the form

$$\frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{xy}}{\partial y} - \frac{\partial p}{\partial x} = 0, \qquad \frac{\partial s_{xy}}{\partial x} - \frac{\partial s_{xx}}{\partial y} - \frac{\partial p}{\partial y} = 0, \tag{8}$$

and the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

provide five equations for the determination of the five unknowns, s_{xx}, s_{xy}, p, u and v.

If $\phi + \pi/4$ is the anticlockwise rotation of the direction of the algebraically greater principal stress in (x, y) plane from the x axis, it follows from (6) that

$$s_{xx} = -k\sin 2\phi, \qquad s_{xy} = k\cos 2\phi. \tag{10}$$

Substituting (10) in (8) yields

$$\cos 2\phi\frac{\partial \phi}{\partial x} + \sin 2\phi\frac{\partial \phi}{\partial y} + \frac{\partial P}{\partial x} = 0, \tag{11}$$

$$\sin 2\phi\frac{\partial \phi}{\partial x} - \cos 2\phi\frac{\partial \phi}{\partial y} + \frac{\partial P}{\partial y} = 0, \tag{12}$$

where $P = p/(2k)$, and substituting (10) in (7) yields

$$2 \sin 2\gamma\left(u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y}\right) + (\sin 2\phi + \sin 2\gamma) \frac{\partial u}{\partial y} + (\sin 2\phi - \sin 2\gamma) \frac{\partial v}{\partial x} + 2 \cos 2\phi \frac{\partial u}{\partial x} = 0, \tag{13}$$

where $\sin 2\gamma = k/\mu$ and the range $0 < k/\mu < 1$ is considered. Let δ denote $(\pi/2) - \gamma$. Since Eq. (13) is not changed when γ is replaced by δ any conclusions remain valid under this replacement.

If Γ is a curve in the (x, y) plane with arc length s and $x'(s) = \cos \zeta$, and $y'(s) = \sin \zeta$, where a prime denotes differentiation with respect to the argument, then along Γ:

$$\frac{\partial \phi}{\partial x} \cos \zeta + \frac{\partial \phi}{\partial y} \sin \zeta = \phi'(s), \tag{14}$$

$$\frac{\partial P}{\partial x} \cos \zeta + \frac{\partial P}{\partial y} \sin \zeta = P'(s), \tag{15}$$

$$\frac{\partial u}{\partial x} \cos \zeta + \frac{\partial u}{\partial y} \sin \zeta = u'(s), \tag{16}$$

$$\frac{\partial v}{\partial x} \cos \zeta + \frac{\partial v}{\partial y} \sin \zeta = v'(s). \tag{17}$$

Eqs. (11), (12), (14) and (15) yield

$$\frac{\partial \phi}{\partial x} \sin 2(\phi - \zeta) = \phi'(s) \sin (2\phi - \zeta) + P'(s) \sin \zeta, \tag{18}$$

$$\frac{\partial \phi}{\partial y} \sin 2(\phi - \zeta) = -\phi'(s) \cos (2\phi - \zeta) - P'(s) \cos \zeta. \tag{19}$$

Characteristic directions, those for which the partial derivatives $\partial\phi/\partial x$ and $\partial\phi/\partial y$ cannot be determined from data given on Γ, are thus given by $\zeta = \phi$ and $\zeta = \phi + \pi/2$. The corresponding curves, for which $dy/dx = \tan \phi$ and $dy/dx = -\cot \phi$, are called α-lines and β-lines respectively. Substitution of $\zeta = \phi$ and $\zeta = \phi + \pi/2$ into (18) or (19) shows that

$$\phi + P = \text{constant along an } \alpha\text{-line},$$

$$\phi - P = \text{constant along a } \beta\text{-line}.$$

These compatibility relations are identical to the Hencky equations for a rigid perfectly-plastic solid.

Suppose now that Γ is neither an α-line nor a β-line. Then the result of eliminating $\partial u/\partial y$, $\partial v/\partial x$ and $\partial v/\partial y$ from (9), (13), (16) and (17) is

$$[\sin 2(\zeta - \phi) - \sin 2\gamma] \frac{\partial u}{\partial x} = -\left[u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y}\right] \sin 2\zeta \sin 2\gamma - 2u'(s) \cos \zeta \sin (\phi + \gamma) \sin (\phi + \delta) + 2v'(s) \sin \zeta \cos (\phi + \gamma) \cos (\phi + \delta). \tag{20}$$

The first term on the right side of (20) may be expressed in terms of $\phi'(s)$ and $P'(s)$ since from (18) and (19).

$$\left[u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y}\right] \sin 2(\phi - \zeta) = \phi'(s)[u \sin (2\phi - \zeta) - v \cos (2\phi - \zeta)] + P'(s)[u \sin \zeta - v \cos \zeta]. \tag{21}$$

It follows from (20) that characteristic directions for the velocity field are given by $\zeta = \phi + \gamma$ and $\zeta = \phi + \delta = \phi + \pi/2 - \gamma$; the corresponding curves, for which $dy/dx = \tan (\phi + \gamma)$ and $dy/dx = -\cot (\phi - \gamma)$, are called α_1-lines and β_1-lines respectively.

The compatibility relation on an α_1-line is obtained by replacing ζ by $\phi + \gamma$ in (20) and using (21) to eliminate $\partial\phi/\partial x$ and $\partial\phi/\partial y$ and is

$$\frac{d}{ds}[u \sin (\phi + \delta) - v \cos (\phi + \delta)] = P'(s)[u \sin (\phi + \gamma) - v \cos (\phi + \gamma)].$$

If U and V are the physical components of the velocity vector in the α_1 and β_1 directions respectively then $U \cos 2\gamma = u \sin (\phi + \delta) - v \cos (\phi + \delta)$ and $V \cos 2\gamma = u \sin (\phi + \gamma) - v \cos (\phi + \gamma)$; consequently Eq. (22) yields the compatibility relation

$$dU/ds + V\, dP/ds = 0 \quad \text{along an } \alpha_1\text{-line.} \tag{22}$$

The analogous result for β_1-lines is obtained by interchanging γ and δ and is

$$dV/ds + U\, dP/ds = 0 \quad \text{along a } \beta_1\text{-line.} \tag{23}$$

For the limiting case of the rigid perfectly-plastic solid $\gamma = 0$, the velocity and stress characteristics coincide and Eqs. (22) and (23) are identical to the familiar Geiringer equations.

References

[1] T. Y. Thomas, *Plastic flow and fracture in solids, Mathematics in Science and Engineering*, **vol. 2**, Academic Press, New York, 1961, p. 94

[2] A. E. Green, *Hypo-elasticity and plasticity*. II, J. Rational Mech. Anal. **5**, 725–734 (1956)

[3] C. Truesdell, *Hypo-elasticity*, J. Rational Mech. Anal. **4**, 83–133 (1955)