A NOTE ON DIPOLAR INERTIA*

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Abstract. With reference to the spatial form of the kinetic energy in the theory of simple dipolar continuum mechanics, a suitable expression is obtained for dipolar inertia. This includes the case in which the coefficient of the kinetic energy due to velocity gradient (in spatial form) may be constant.

Using a system of generalized coordinates q_α ($\alpha = 1, \cdots, \nu$) for generalized continuum mechanics, Green and Rivlin [1] adopted a kinetic energy per unit mass in the form

$$T = \tfrac{1}{2} \sum_{\alpha=1}^{\nu} a_{\alpha\beta}\dot{q}_\alpha \dot{q}_\beta \,, \tag{1}$$

where $a_{\alpha\beta}$ is a function of q_γ ($\gamma = 1, \cdots, \nu$) and a superposed dot denotes differentiation with respect to time t. It was then shown that the material time derivative of T is given by

$$\dot{T} = \sum_{\alpha=1}^{\nu} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha}\right]\dot{q}_\alpha \,, \tag{2}$$

and that the inertia term corresponding to the velocity $\dot{q}_\alpha$ is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha}. \tag{3}$$

Taking the reference frame to be rectangular Cartesian and using tensor notation, in the theory of simple dipolar continuum mechanics in which the basic generalized velocities are v_i and $v_{i,j}$, a suitable form for the kinetic energy per unit mass is

$$\tfrac{1}{2}v_i v_i + T, \tag{4}$$

where

$$T = \tfrac{1}{2} m_{kl} v_{i,k} v_{i,l} \,, \tag{5}$$

and $(\ )_{,i}$ denotes partial differentiation with respect to the spatial coordinate x_i holding t fixed. Also, m_{kl} is a symmetric function of displacement gradients $x_{i,A}$ where $(\ )_{,A}$ denotes partial differentiation with respect to a reference position X_A holding t fixed. If we assume that T is unaltered by a *static* rigid body rotation, then

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$$m_{kl} = x_{k,A}x_{l,B}M_{AB}(E_{PQ}), \tag{6}$$

where M_{AB} is symmetric and

$$E_{PQ} = x_{i,P}x_{i,Q} . \tag{7}$$

The expression (5) for T is not in the Lagrangian form (1) so that we cannot immediately apply the result (3) to deduce the correct inertia term corresponding to the velocity gradient $v_{i,j}$. However, from (5) and (6) we have

$$T = T^* \text{ (say)} = \tfrac{1}{2}M_{AB}v_{i,A}v_{i,B} . \tag{8}$$

We may now take $x_{i,A}$ to be a set of generalized coordinates so that the inertia term corresponding to $v_{i,A}$ is f_{Ai} with

$$\dot{T}^* = f_{Ai}v_{i,A} , \tag{9}$$

$$f_{Ai} = \frac{d}{dt}\left(\frac{\partial T^*}{\partial v_{i,A}}\right) - \frac{\partial T^*}{\partial x_{i,A}}. \tag{10}$$

When M_{AB} is constant then (10) reduces to

$$f_{Ai} = M_{AB}\dot{v}_{i,B} . \tag{11}$$

We now wish to find the inertia term corresponding to the spatial form (5) of the kinetic energy. Since

$$T(x_{i,A} , v_{i,A}) = T^*(x_{i,A} , v_{i,j}), \tag{12}$$

a straightforward calculation yields

$$f_{Ai} = f_{ji}(\partial X_A/\partial x_j), \tag{13}$$

where

$$f_{ji} = \frac{d}{dt}\left(\frac{\partial T^*}{\partial v_{i,j}}\right) - \frac{\partial T^*}{\partial v_{i,k}}v_{j,k} + \frac{\partial T^*}{\partial v_{k,j}}v_{k,i} - \frac{\partial T^*}{\partial x_{i,B}}x_{j,B} . \tag{14}$$

Also

$$\dot{T}^* = \dot{T} = f_{Ai}v_{i,A} = f_{ji}v_{i,j} , \tag{15}$$

and we adopt f_{ji} as the inertia term corresponding to $v_{i,j}$.

In the special case when

$$m_{kl} = d^2\,\delta_{kl} , \tag{16}$$

where d is a constant and δ_{kl} is the Kronecker symbol, the inertia term (14) reduces to

$$f_{ji} = d^2[\dot{\overline{v_{i,j}}} - v_{i,k}v_{j,k} + v_{k,j}v_{k,i}] = d^2[(\dot{v}_i)_{,j} - v_{i,k}v_{k,j} - v_{i,k}v_{j,k} + v_{k,j}v_{k,i}]. \tag{17}$$

The above inertia term differs from

$$d^2\,\dot{\overline{v_{i,j}}}$$

which is the one used by Bleustein and Green [2]. However, for the particular problem of flow along a circular channel considered by Bleustein and Green, the difference does not affect the velocity field—it only changes the value of the pressure.

Inertia terms for other types of multipolar continuum mechanics can be discussed in a similar manner. It should be noted that when M_{AB} in (6) is constant (or depends only on X_A), then the inertia coefficient m_{kl} satisfies Eringen's [3] equation for micro-inertia coefficients, but this is only a special case of the above.

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References

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