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A GENERAL ONE-PHASE STEFAN PROBLEM*

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1. Introduction. In this paper we discuss the problem of determining $u(x, t)$ and the free boundary $s(t) > 0$ subject to the conditions

$$\begin{gathered} u_{xx} = u_t, \quad 0 < x < s(t); \quad u(x, 0) = \varphi(x), \quad u_x(0, t) = f(t), \\ u(s(t), t) = g(t), \quad -\lambda(t)s'(t) + u_x(s(t), t) = q(t), \quad s(0) = a. \end{gathered} \tag{1.1}$$

Here f, g, λ, q are defined for all $t \geq 0$, and φ and a are the given data of the problem, with $\varphi(a) = g(0)$, $a > 0$, and $\lambda(t) > 0$. When

$$g(t) \equiv 0, \quad q(t) \equiv 0, \quad \lambda(t) = \lambda = \text{constant}, \quad \varphi(x) \leq 0, \quad f(t) \geq 0, \tag{1.2}$$

then (1.1) has a simple interpretation as a problem in heat conduction with melting: the region $0 \leq x \leq a$ is initially solid with the temperature distribution $\varphi(x)$, the region $a < x < \infty$ is liquid at the melting temperature 0, there is flux $f(t)$ directed out of the solid at the fixed face $x = 0$, and $\lambda = \rho l/k$ where ρ is the common density of liquid and solid, l is the latent heat, and k the coefficient of thermal conductivity. $u(x, t)$ is the temperature in the solid part and $s(t)$ the position of the interface. With the restrictions (1.2) on the data it has been proved (when regularity conditions are imposed on the data) that $u(x, t)$ and $s(t)$ exist for all t, are unique, depend continuously on the data, and that $s'(t) \geq 0$ [2], [3], [5], [6], [7], [8], [9]. In these papers the region $0 \leq x \leq a$ is liquid and $a < x < \infty$ is solid so the inequalities on φ and f are reversed and $-\lambda$ is replaced by λ in the free boundary condition. The case $a = 0$, subject to (1.2), is treated in [2], [3], [5], [7], [12].

If in (1.2) we remove the restriction $q(t) \equiv 0$ and allow $q(t)$ to be an arbitrary continuous function

$$g(t) \equiv 0, \quad \lambda(t) = \lambda = \text{constant}, \quad \varphi(x) \leq 0, \quad f(t) \geq 0, \tag{1.3}$$

then existence, uniqueness, and continuous dependence are proved in [10], [11] when regularity conditions are imposed on the data. The condition $q(t) \geq 0$ is imposed in [10] but this restriction is unnecessary and plays no role in the proofs. The interpretation of (1.1) as a heat conduction problem with melting is as above, with $q(t)$ a flux situated in the moving interface which may be directed into ($q(t) > 0$) or out of ($q(t) < 0$) the solid. The free boundary $s(t)$ is not, in general, a monotonic function. Complete melting may occur in a finite time σ, where σ is the smallest value satisfying $\lambda a + H(\sigma) = 0$, where

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$$H(t) = \int_0^t [f(\tau) - q(\tau)]\, dt - \int_0^a \varphi(x)\, dx. \tag{1.4}$$

If there is no t such that $\lambda a + H(t) = 0$ then complete melting does not occur in a finite time; we then write $\sigma = \infty$. Thus the solution of (1.1) exists in the case (1.3) to $t = \sigma$.

The existence and uniqueness proof for (1.1) in the case (1.3) is divided into two parts, first a proof of local existence and uniqueness and then the extension of that solution to $t = \sigma$. The proof of the local theorem does not make use of the hypotheses on the signs of $f(t)$ and $\varphi(x)$; only the regularity properties of these functions are used. Accordingly it seems plausible that, with appropriate regularity conditions in the data, we can prove a local existence and uniqueness theorem for (1.1) with no restrictions on the signs of the data and with the requirement $g(t) \equiv 0$ removed. This is done in Sec. 2. Next we define σ to be the supremum of those t for which (1.1) has a solution. For (1.2) we know that $\sigma = \infty$ and for (1.3) we know that $\sigma = \infty$ or $s(\sigma) = 0$ with $s(t) > 0$ for $0 < t < \sigma$. The question then arises as to whether, for (1.1), it is possible for σ to be finite and for $s(t)$ not to approach 0 as $t \to \sigma$. We show in Sec. 2 that there is indeed this third possibility and that for such a σ $\liminf u_x(s(t), t) = -\infty$ as $t \to \sigma$. It follows that $\liminf s'(t) = -\infty$ as $t \to \sigma$. We show by an example that this third possibility does occur. These three possibilities describe completely the possible behavior at $t = \sigma$.

If in (1.1) we delete the term $-\lambda(t)s'(t)$ then we get a free boundary problem for the heat equation in which Cauchy data are prescribed on the free boundary. Wentzel [14] has discussed a problem of this sort. The following problem arises in statistical decision theory:

$$u_{xx} = u_t, \quad 0 < x < s(t); \quad u_x(0, t) = 1/2, \quad s(0) = 0,$$
$$u(s(t), t) = 1/2t, \quad u_x(s(t), t) = 0. \tag{1.5}$$

The problem discussed by Chernoff in [1] can be reduced to (1.5). We may consider the possibility of obtaining existence theorems for this general class of problems by adding the term $-\lambda s'(t)$, λ a parameter, to the flux condition at the free boundary and then letting $\lambda \to 0$. We have elsewhere [13] obtained some results in this direction.

2. Existence and uniqueness of a solution of (1.1). We will suppose the data of (1.1) satisfy the following regularity conditions: $f(t)$, $g'(t)$, $\lambda(t)$ are continuous on $t \geq 0$ and $\varphi'(x)$ is continuous on $0 \leq x \leq a$. We suppose that $\lambda(t) > 0$ and that $\varphi(a) = g(0)$. The regularity conditions are not minimal but they are appropriate for the fixed point method of proof we use here. We note again that $a > 0$.

We define a solution $u(x, t)$, $s(t) > 0$ of (1.1) for $0 \leq t < T$, where $0 < T \leq \infty$, as follows [6, p. 216]: (a) u_{xx} and u_t are continuous for $0 < x < s(t)$, $0 < t < T$; (b) u and u_x are continuous for $0 \leq x \leq s(t)$, $0 < t < T$; (c) u is continuous also for $t = 0$, $0 < x \leq a$ and u is bounded at $(0, 0)$; (d) $s'(t)$ exists and is continuous on $0 \leq t < T$; (e) (1.1) is satisfied. To define a solution of (1.1) for $0 \leq t \leq T$ the $<T$ inequalities in (a), (b), (d), and (e) are replaced by $\leq T$.

THEOREM. *For sufficiently small T* (1.1) *has a unique solution. Let σ be the supremum of those values of T for which* (1.1) *has a solution. Then the solution for $0 \leq t < \sigma$ is unique and there are three possibilities*: (a) $\sigma = \infty$; (b) *σ is finite and $s(\sigma-) = 0$*; (c) *σ is finite, $\liminf u_x(s(t), t) = -\infty$ as $t \to \sigma$, and $s(t)$ does not tend to 0 as $t \to \sigma$.*

To prove the theorem we introduce the fundamental solution of the heat equation, K, and the Green and Neumann functions G and N for the upper half plane $t > 0$ [6]:

$$\begin{aligned} K(x, t; \xi, \tau) &= [4\pi(t-\tau)]^{-1/2} \exp[-(x-\xi)^2/4(t-\tau)], \\ G(x, t; \xi, \tau) &= K(x, t; \xi, \tau) - K(-x, t; \xi, \tau), \\ N(x, t; \xi, \tau) &= K(x, t; \xi, \tau) + K(-x, t; \xi, \tau). \end{aligned}$$

Let $u(x, t)$, $s(t)$ be a solution of (1.1) for $0 \le t \le T$. Integrating the identity

$$\frac{\partial}{\partial \xi}(Nu_\xi - uN_\xi) - \frac{\partial}{\partial \tau}(Nu) = 0$$

over the domain $0 < \xi < s(\tau)$, $\epsilon < \tau < t - \epsilon$, using Green's theorem, and letting $\epsilon \to 0$, we get

$$\begin{aligned} u(x, t) = &\int_0^t [u_\xi(s(\tau), \tau) + g(\tau)s'(\tau)]N(x, t; s(\tau), \tau)\, d\tau \\ &- \int_0^t g(\tau)N_\xi(x, t; s(\tau), \tau)\, d\tau \\ &- \int_0^t f(\tau)N(x, t; 0, \tau)\, d\tau + \int_0^a \varphi(\xi)N(x, t; \xi, 0)\, d\xi. \end{aligned} \tag{2.1}$$

In (2.1) we replace N_ξ by $-G_x$ and then differentiate with respect to x on both sides. The G_{xx} which appears in the second term on the right may be replaced by $G_t = -G_\tau$. We have

$$\frac{d}{d\tau} G(x, t; s(\tau), \tau) = G_\xi(x, t; s(\tau), \tau)s'(\tau) + G_\tau(x, t; s(\tau), \tau). \tag{2.2}$$

Using (2.2) and replacing N_x by $-G_\xi$ in the last term on the right (in the differentiated version of (2.1)) we get, on performing partial integrations, using $\varphi(a) = g(0)$, and writing $v(t) = u_x(s(t), t)$

$$\begin{aligned} u_x(x, t) = &\int_0^t v(\tau)N_x(x, t; s(\tau), \tau)\, d\tau + \int_0^t g'(\tau)G(x, t; s(\tau), \tau)\, d\tau \\ &- \int_0^t f(\tau)N_x(x, t; 0, \tau)\, d\tau + \int_0^a \varphi'(\xi)G(x, t; \xi, 0)\, d\xi. \end{aligned} \tag{2.3}$$

In (2.3) we let $x \to s(t)$. Using Lemma 1 of [4] we get

$$\begin{aligned} v(t) = 2&\int_0^t v(\tau)N_x(s(t), t; s(\tau), \tau)\, d\tau + 2\int_0^t g'(\tau)G(s(t), t; s(\tau), \tau)\, d\tau \\ - 2&\int_0^t f(\tau)N_x(s(t), t; 0, \tau)\, d\tau + 2\int_0^a \varphi'(\xi)G(s(t), t; \xi, 0)\, d\xi. \end{aligned} \tag{2.4}$$

We have also the equation

$$s(t) = a - \int_0^t (\lambda(\tau))^{-1}q(\tau)\, d\tau + \int_0^t (\lambda(\tau))^{-1}v(\tau)\, d\tau. \tag{2.5}$$

We must show now that a solution $v(t)$ and $s(t) > 0$ of (2.4) and (2.5) continuous on $0 \le t \le T$ gives a solution of (1.7) by defining $u(x, t)$ by (2.1), with $u_\xi(s(\tau), \tau)$ replaced by $v(\tau)$. The proof of this parallels the argument in [10, p. 57] except that in equation (12) of [10] we must replace $u(s(\tau), \tau)$ by $u(s(\tau), \tau) - g(\tau)$.

To prove the existence of a solution of (2.4) and (2.5) let $C(T)$ be the Banach space of continuous functions on $0 \le t \le T$ with maximum norm. Let $C(T, M)$ be the closed sphere $\|v\| \le M$. If we write $v = Sv$ for (2.4) then $w = Sv$, with $s(t)$ defined by (2.5), defines a mapping of $C(T, M)$ into $C(T)$. We show that we can choose T and M so that S is a contracting mapping of $C(T, M)$ into itself. Following the argument in [10, p. 57–60] we see that S is a mapping of $C(T, M)$ into itself if $M = |\varphi'(a)| + 1$ and T is subject to the inequalities (16) and (20) of [10], where in (20) we must insert the term $2\,\|g'\|\,T$ on the left (also $k = 1$ and $\alpha = \|\lambda^{-1}\|$ in those inequalities). If, furthermore, inequalities (28) of [10] and $F(M, T) + \|g'\| f(M, T) < \frac{1}{2}$ are satisfied, where $F(M, T)$ is defined in [10, p. 60] and $f(M, T)$ goes to 0 with T, then S is a contracting mapping of $C(T, M)$ into itself. Thus there is a unique fixed point, i.e., (2.4) and (2.5) have solutions on $0 \leq t \leq T$ which is unique subject to $\|v\| \leq M$. That the uniqueness is independent of this condition is proved as in [10, p. 61].

We have proved above the existence of a local solution of (1.1), i.e., that $\sigma > 0$. That the solution of (1.1) is unique on $0 \leq t < \sigma$ is proved the same way as uniqueness for the local solution. To prove the final part of the theorem we note there are the following possibilities, if σ is finite, for $v(t)$ as $t \to \sigma$:

$$\begin{aligned} &\text{(a)} \quad v(t) \text{ tends to a finite limit,} \\ &\text{(b)} \quad v(t) \to +\infty, \\ &\text{(c)} \quad \liminf v(t) \text{ is finite, } \limsup v(t) = +\infty, \\ &\text{(d)} \quad v(t) \text{ is bounded,} \\ &\text{(e)} \quad \liminf v(t) = -\infty. \end{aligned} \tag{2.6}$$

Throughout the ensuing discussion we may assume that, if $s(t)$ has a finite limit, that limit is positive, for otherwise we are in case (b) of the theorem. Suppose (2.6a) is true. Then $s(t)$ also has a finite limit as $t \to \sigma$. For $0 < t < \sigma$ we may consider the inequalities specified in the previous paragraph, where $a = s(t)$, $\varphi(x) = u(x, t)$, $M(t) = |v(t)| + 1$, and norms of functions of t are taken over the interval from t to $t + T$. Let $T^*(t)$ be the supremum of those $T(t)$ satisfying the inequalities. Then $T^*(t) > 0$ and, since $v(\sigma-)$ exists, $T^*(\sigma-)$ exists and is positive. We may then choose t sufficiently close to σ so that $T^*(t) > \sigma - t$. This implies that the solution of (1.1) can be extended past σ, a contradiction. Thus (2.6a) cannot be true. (2.6b) implies that $v(t)$ is positive in the vicinity of σ and therefore bounded (by the argument in [10, p. 62]), a contradiction. (2.6d) implies, again by the argument in [10, p. 62], that $v(t)$ has a finite limit as $t \to \sigma$ so that the solution can be extended past σ; thus (2.6d) is also ruled out. It remains to eliminate (2.6c). Let $m = \inf v(t)$ on $0 \leq t \leq \sigma$ and let $v^*(t) = v(t) - m$, $q^*(t) = q(t) - m$. Then $v^*(t) \geq 0$. From

$$s(t) = a - \int_0^t (\lambda(\tau))^{-1} q^*(\tau)\, dt + \int_0^t (\lambda(\tau))^{-1} v^*(\tau)\, dt$$

we see that either $s(\sigma-)$ is finite or $s(\sigma-) = +\infty$. Analogous to (2.4) we may write, where $\mu < \sigma$ is to be determined and $\sigma - \mu \leq t < \sigma$,

$$\begin{aligned} v^*(t) = &-m + 2\int_{\sigma-\mu}^t v^*(\tau) N_x(s(t), t; s(\tau), \tau)\, d\tau + 2m \int_{\sigma-\mu}^t N_x(s(t), t; s(\tau), \tau)\, d\tau \\ &+ 2\int_{\sigma-\mu}^t g'(\tau) G(s(t), t; s(\tau), \tau)\, d\tau - 2\int_{\sigma-\mu}^t f(\tau) N_x(s(t), t; 0, \tau)\, d\tau \\ &+ 2\int_0^{s(\sigma-\mu)} u_x(\xi, \sigma - \mu) G(s(t), t; \xi, \sigma - \mu)\, d\xi. \end{aligned} \tag{2.7}$$

The second term, the last, and the next to last term on the right of (2.7) may be estimated by the methods of [10, p. 62]. Let $\|v^*\|$ and $\|u_x\|$ be respectively the maximum on $\sigma - \mu \leq \tau \leq t$ of $v^*(\tau)$ and on $0 \leq x \leq s(\sigma - \mu)$ of $|u_x(x, \sigma - \mu)|$. The norms of other functions of t will be taken over $0 \leq t \leq \sigma$. Then the second term is $\leq \mu^{1/2}\, \|\lambda^{-1}\|\, \|v^*\|\, \|q^*\|$, the last term is $\leq 4\,||u_x||$, and the next to last term is $\leq ||f||$. The term involving $g'(\tau)$ is $\leq 4\mu^{1/2}\,||g'||$. The third term on the right of (2.7) is the sum of two terms, of which the first is

$$m \int_{\sigma-\mu}^{t} [-(s(t) - s(\tau))/(t - \tau)]K(s(t), t; s(\tau), \tau)\, d\tau$$
$$= m \int_{\sigma-\mu}^{t} \left[\int_{\tau}^{t} \lambda^{-1}(\zeta)q^*(\zeta)\, d\zeta - \int_{\tau}^{t} \lambda^{-1}(\zeta)v^*(\zeta)\, d\zeta\right](t - \tau)^{-1}K(s(t), t; s(\tau), \tau)\, d\tau$$
$$\leq \mu^{1/2}\, |m|\, ||\lambda^{-1}||\, (||q^*|| + ||v^*||,$$

and the second is

$$(m/2\pi^{1/2}) \int_{\sigma-\mu}^{t} -(s(t) + s(\tau))/(t - \tau)^{3/2} \exp[-(s(t) + s(\tau))^2/4(t - \tau)]\, d\tau. \quad (2.8)$$

If $m \geq 0$ then (2.8) has the upper bound 0. If $m < 0$ and $s(t)$ has a finite positive limit at σ then, letting $a(\mu)$ and $b(\mu)$ be, respectively, the maximum and minimum of $s(\tau)$ on $\sigma - \mu \leq t \leq \sigma$, (2.8) is less than or equal to

$$|m|\, \pi^{-1/2} \int_{\sigma-\mu}^{t} a(t - \tau)^{-3/2} \exp(-b^2/(t - \tau))\, d\tau$$
$$= 2\, |m|\, a/b\pi^{1/2} \int_{b/(t-\sigma+\mu)^{-1/2}}^{\infty} \exp(-\xi^2)\, d\xi < |m|\, a/b.$$

Here $b(\mu) > 0$. If $m < 0$ and $s(t) \to +\infty$ as $t \to \sigma$ then for sufficiently small μ $s(t) + s(\tau) < (s(t) + s(\tau))^2$ and, using $xe^{-x} \leq e^{-1}$ for $x > 0$, (2.8) is less than

$$2\, |m|/e\pi^{1/2} \int_{\sigma-\mu}^{t} (t - \tau)^{-1/2}\, d\tau < |m|\, \mu^{1/2}.$$

Thus

$$0 \leq v^*(t) \leq |m| + \mu^{1/2}\, ||\lambda^{-1}||\, ||v^*||\, ||q^*|| + ||f|| + 4||u_x||$$
$$+ 4\mu^{1/2}\, ||g'|| + \mu^{1/2}\, |m|\, ||\lambda^{-1}||\, (||q^*|| + ||v^*||) + F(m, \mu), \quad (2.9)$$

where F is the maximum of $|m|\, a(\mu)/b(\mu)$ and $|m|\, \mu^{1/2}$. From (2.9) we get

$$[1 - \mu^{1/2}\, ||\lambda^{-1}||\, (||q^*|| + |m|)]\, ||v^*|| \leq |m| + ||f|| + 4||u_x||$$
$$+ 4\mu^{1/2}\, ||g'|| + \mu^{1/2}\, |m|\, ||\lambda^{-1}||\, ||q^*|| + F(m, \mu). \quad (2.10)$$

We select μ so that the bracket on the left of (2.10) is positive. Then (2.10) implies that $v^*(t)$ is bounded as $t \to \sigma$ so that (2.6c) leads to a contradiction. Thus the only possibility is (2.6e).

We show now by an example that the third possibility occurs:

$$u_{xx} = u_t, \quad 0 < x < s(t); \quad u(x, 0) = \varphi(x), \quad u_x(0, t) = 0,$$
$$u(s(t), t) = 0, \quad -\lambda s'(t) + u_x(s(t), t) = 0, \quad s(0) = a. \quad (2.11)$$

Here $\varphi(x) \geq 0$, $\varphi(a) = 0$. We may suppose the solution reflected across the t axis. Since $u(\pm s(t), t) = 0$ and $u(x, 0) = \varphi(x) \geq 0$ we conclude from the maximum principle that $u(x, t) \geq 0$. Thus $u_x(s(t), t) \leq 0$ and therefore $s'(t) \leq 0$. When $g(t) \equiv 0$ and $\lambda(t) = \lambda =$ constant we get, by an application of Green's theorem to (1.1) [10, p. 55],

$$s(t) = a + \lambda^{-1}H(t) + \lambda^{-1} \int_0^{s(t)} u(x, t)\, dx.$$

For (2.11) we get then

$$\lambda a - \int_0^a \varphi(x)\, dx = \lambda s(t) - \int_0^{s(t)} u(x, t)\, dx. \quad (2.12)$$

Suppose the left side of (2.12) is negative; then we can eliminate cases (a) and (b) of the theorem. In case (b), σ is finite and $s(\sigma) = 0$, we have only to let $t \to \sigma$ to get a contradiction ($u(x, t)$ is bounded by the maximum principle). Suppose we have case (a): a solution exists for all t. We define u^0 to be the solution of

$$u_{xx} = u_t, \quad -a < x < a, \quad t > 0; \quad u(\pm x, 0) = \varphi(x), \quad u(\pm a, t) = 0.$$

Then since $u^0(\pm s(t), t) > 0$ and u^0 coincides with u at $t = 0$ we have, by the maximum principle, $u^0 \geq u \geq 0$. Thus

$$0 \leq \int_0^{s(t)} u(x, t)\, dx \leq \int_0^{s(t)} u^0(x, t)\, dx < \int_0^a u^0(x, t)\, dx. \tag{2.13}$$

The right side of (2.13) goes to 0 as $t \to \infty$. Letting $t \to \infty$ in (2.12) we get

$$\lambda a - \int_0^a \varphi(x)\, dx = \lambda s(\infty),$$

a contradiction since $s(\infty) \geq 0$. Thus case (c) of the theorem applies in this example. Since $s'(t) \leq 0$ it is clear that $s(\sigma-)$ exists and is positive.

We have noted that case (c) of the theorem does not hold when conditions (1.3) are satisfied. We may also exclude possibility (c) in the case $f(t) \geq 0$, $\varphi(x) \leq \varphi(a)$, and $g'(t) \geq 0$. Then $u(x, t) \leq g(t)$; this follows from Lemma 1 of [12] applied to $g(T) - u(x, t)$ for an arbitrary fixed T and for $0 \leq t \leq T$. Thus $v(t) \geq 0$ and we cannot have possibility (c). We may interpret $0 < x < s(t)$ as solid and $x \geq s(t)$ as liquid, where the melting temperature $g(t)$ is nondecreasing, heat is withdrawn at the rate $f(t)$ at the fixed face, and there is flux $q(t)$ at the moving interface.

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