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A PERTURBATION METHOD FOR BOUNDARY-VALUE PROBLEMS IN DYNAMIC ELASTICITY, PART II*

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I. Introduction. In a recent paper [1, hereafter referred to as Part I], the authors presented a perturbation method for treating problems of steady-state elastic waves in which the shear and compressional wave numbers, always distinct in an actual elastic solid, are replaced by their root-mean-square value and a small dimensionless parameter. The total wave fields are then expanded in terms of this parameter into successive orders of perturbed fields. The chief advantage of this method is that the equations governing all orders of perturbation contain the common rms wave number. Thus, in the first few orders, solutions can be generated with nearly the same ease as for corresponding scalar wave problems.

The previous development and applications of the perturbation method in Part I and in [2] started with the equations of dynamic elasticity in terms of the Lamé displacement potentials. However, for treating problems with displacement boundary conditions, a more convenient perturbation scheme can be employed which starts directly with the equations of motion in terms of the displacements. A brief description of this was begun in the final section of Part I in connection with the diffraction of plane waves by a semi-infinite rigid ribbon; but even there the potentials were used concurrently. Here we present a perturbation method involving only the displacement components.

After developing the perturbation equations for the various orders of displacements in the following section, we discuss in detail the perturbation of plane waves in Sec. III. For problems involving scattering of plane waves, the results of Sec. IV establish the proper radiation conditions for the scattered wave at infinity in each order of perturbation. In addition a uniqueness theorem is proved for the nth order scattered wave. In Sec. V, this new scheme is applied to the diffraction of plane elastic waves by a semi-infinite clamped strip. This problem has been treated by Roseau [3] and in Part I. Unfortunately, because the first order perturbation of the incident wave was incomplete, the result in Part I was in error. The correct results are presented herewith. Finally, we construct a two-term perturbation solution for the diffraction of plane elastic waves by a rigid strip of finite width. Ang and Knopoff [4] and Harumi [5] have treated this problem

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with emphasis in the far field at low frequencies. Our perturbation solution yields results which are most accurate in the near field.

II. Perturbation of the displacement equations of motion. In terms of the characteristic wave speeds c_1 (P-wave) and c_2 (S-wave), the equations of motion for an isotropic, homogeneous, elastic solid are given by

$$(\kappa^2 - 1)\nabla\nabla\cdot\boldsymbol{u} + \nabla^2\boldsymbol{u} = (\omega^2/c_2^2)\boldsymbol{u} \tag{1}$$

where

$$\kappa = c_1/c_2 = [(2 - 2\nu)/(1 - 2\nu)]^{1/2}$$

is the ratio of the wave speeds and ν is the Poisson's ratio of the solid. Only steady-state motion is considered here and the time factor exp $(-i\omega t)$ is omitted from the displacement vector $\boldsymbol{u} = (u, v, w)$.

Taking $k_1 = \omega/c_1$, $k_2 = \omega/c_2$ and introducing the rms wave number k with the formulas $k_1 = k(1 - 2\epsilon)^{1/2}$, $k_2 = k(1 + 2\epsilon)^{1/2}$ and $\epsilon = (k_2^2 - k_1^2)/2(k_2^2 + k_1^2)$ as in Eqs. (5a) and (6a) of Part I, the above equation is rewritten as

$$4\epsilon\nabla\nabla\cdot\boldsymbol{u} + (1 - 2\epsilon)\nabla^2\boldsymbol{u} + k^2(1 - 4\epsilon^2)\boldsymbol{u} = 0. \tag{2}$$

Assuming perturbations series for $\boldsymbol{u}$ and the stress dyadic $\boldsymbol{\tau}$

$$\boldsymbol{u} = \sum_{n=0}^{\infty} \epsilon^n\boldsymbol{u}^{(n)}, \qquad \boldsymbol{\tau} = \sum_{n=0}^{\infty} \epsilon^n\boldsymbol{\tau}^{(n)}$$

yields the following equation for the nth-order displacement vector:

$$(\nabla^2 + k^2)\boldsymbol{u}^{(n)} = 2\nabla^2\boldsymbol{u}^{(n-1)} - 4\nabla\nabla\cdot\boldsymbol{u}^{(n-1)} + 4k^2\boldsymbol{u}^{(n-2)} \tag{3}$$

where it is understood that $\boldsymbol{u}^{(n)} = 0$ for $n < 0$.

The nth-order stresses and displacements are related through Hooke's Law

$$\boldsymbol{\tau}^{(n)} = \mu[(\kappa^2 - 2)\nabla\cdot\boldsymbol{u}^{(n)}\boldsymbol{I} + \nabla\boldsymbol{u}^{(n)} + \boldsymbol{u}^{(n)}\nabla] \tag{4}$$

where $\boldsymbol{I}$ is the idemfactor and μ is the shear modulus of the material. Note that κ^2 is not expanded in ϵ in the stress-displacement relation.

The equation for the zeroth-order displacement vector becomes the homogeneous Helmholtz equation with the root mean square wave number k:

$$(\nabla^2 + k^2)\boldsymbol{u}^{(0)} = 0. \tag{5}$$

It consists of three uncoupled equations for the Cartesian components of $\boldsymbol{u}^{(0)}$. Consequently, for problems in which $\boldsymbol{u}$ is specified on the boundary, the zeroth-order solution for each component is identical with that for acoustic waves with analogous boundary conditions.

From Eq. (3) the first-order field equation follows as

$$(\nabla^2 + k^2)\boldsymbol{u}^{(1)} = 2\nabla^2\boldsymbol{u}^{(0)} - 4\nabla\nabla\cdot\boldsymbol{u}^{(0)} \tag{6}$$

which has a particular solution

$$\boldsymbol{u}_p^{(1)} = \boldsymbol{r}\cdot\nabla\boldsymbol{u}^{(0)} - 2\boldsymbol{r}\cdot\boldsymbol{u}^{(0)}\nabla \tag{7}$$

in dyadic notation. The complete solution of Eq. (6) is given by Eq. (7) plus a complementary solution, i.e.,

$$u^{(1)} = u_p^{(1)} + u_c^{(1)}. \tag{8}$$

In the second-order perturbation, the solution will depend on both the zeroth- and first-order solutions, since from (3)

$$(\nabla^2 + k^2)u^{(2)} = 2\nabla^2 u^{(1)} - 4\nabla\nabla \cdot u^{(1)} + 4k^2 u^{(0)}. \tag{9}$$

Utilizing Eqs. (5) through (8), we can manipulate Eq. (9) into

$$(\nabla^2 + k^2)u^{(2)} = 2\nabla^2(u_c^{(1)} + 4u^{(0)}) - 4\nabla\nabla\cdot(u_c^{(1)} + 2u^{(0)}) - 2k^2 r\cdot(\nabla u^{(0)} + 2u^{(0)}\nabla) - 4r\cdot\nabla(\nabla\cdot u^{(0)})\nabla. \tag{10}$$

It can be shown after some lengthy, but straightforward, calculation that a particular solution of the above is

$$u_p^{(2)} = r\cdot\nabla[u_c^{(1)} + (2 - n/2)u^{(0)}] - 2r\nabla\cdot(u_c^{(1)} + u^{(0)}) - \tfrac{1}{2}k^2 r^2 u^{(0)} - 2k^2 r(r\cdot u^{(0)}) - 2r[r\cdot\nabla(\nabla\cdot u^{(0)})] \tag{11}$$

where n = number of space dimensions.

We shall not consider the second- and higher-order perturbations further. From previous investigations [1], [2] it appears that a two-term perturbation solution is generally accurate in calculating the near field at low frequencies for diffractions of elastic waves.

Eqs. (3) through (11) above can be compared with Eqs. (8) through (17) in Part I.

III. Perturbation of plane waves. For the analysis of diffraction of plane waves by an obstacle, the solution for each order of perturbation must contain explicitly the "nth-order incident wave" which is defined as the coefficient of ϵ^n in the MacLaurin series in ϵ for the given incident wave. The total nth-order solution minus the nth-order incident wave will be known as the "nth-order scattered wave." In this section we shall examine in detail the expansion of plane elastic waves in a series in ϵ. In Sec. IV we shall derive the appropriate radiation conditions for the nth-order scattered waves and prove such conditions are sufficient to assure a unique solution to the "nth-order problem."

An incident (superscript i) compressional and shear wave (subscripts A and B, respectively) are represented by

$$u_A^{(i)} = Al \exp(ik_1 r\cdot l), \qquad u_B^{(i)} = Bm \exp(ik_2 r\cdot n) \tag{12}$$

where l and n are the unit wave normals or propagation vectors, m is a unit vector perpendicular to n, and A and B are constants. The two waves are combined to form the incident field $u^{(i)} = u_A^{(i)} + u_B^{(i)}$.

To determine the perturbation of these plane waves, we first expand $k_{2,1} = k(1 \pm 2\epsilon)^{1/2}$ and then obtain

$$u^{(i)} = Al \exp(ikr\cdot l)[1 - ik\epsilon r\cdot l + O(\epsilon^2)] + Bm \exp(ikr\cdot n)[1 + ik\epsilon r\cdot n + O(\epsilon^2)]. \tag{13}$$

Hence, the zeroth- and first-order perturbations of $u^{(i)}$ become

$$u_A^{(0)(i)} = Al \exp(ikr\cdot l), \qquad u_B^{(0)(i)} = Bm \exp(ikr\cdot n) \tag{14}$$

$$u_A^{(1)(i)} = -ikr\cdot l u_A^{(0)(i)}, \qquad u_B^{(1)(i)} = ikr\cdot n u_B^{(0)(i)} \tag{15}$$

with $u^{(n)(i)} = u_A^{(n)(i)} + u_B^{(n)(i)}$.

In an actual diffraction problem involving plane incident waves, the solutions of the perturbed equations of motion must contain explicitly the expressions (14) and (15). Since the waves in Eq. (14) are also solutions of Eq. (5), the zeroth-order solution naturally will contain them. On the other hand, when Eqs. (15) are substituted into the first-order particular solution (7), we obtain

$$\boldsymbol{u}_p^{(1)(i)} = -ikr\cdot \boldsymbol{l}\boldsymbol{u}_A^{(0)(i)} + ikr\cdot \boldsymbol{n}\boldsymbol{u}_A^{(0)(i)} - 2\boldsymbol{r}\cdot \boldsymbol{u}_B^{(0)(i)}\nabla \tag{16}$$

which differs from the sum of Eqs. (15) by the last term on the right. However, since $\boldsymbol{n}\cdot\boldsymbol{m} = 0$, it can be checked that for plane shear waves $\boldsymbol{r}\cdot\boldsymbol{u}_B^{(0)(i)}\nabla$ satisfies the homogeneous Helmholtz equation and so can be separated explicitly from $\boldsymbol{u}_c^{(1)}$, leaving another complementary solution in (8). The same is not true for $\boldsymbol{r}\cdot\boldsymbol{u}_A^{(0)(i)}\nabla$.

Thus, for application to diffraction problems of incident plane waves, we modify the first-order particular solutions as

$$\boldsymbol{u}_p^{(1)} = \boldsymbol{r}\cdot\nabla\boldsymbol{u}^{(0)} - 2\boldsymbol{r}\cdot\boldsymbol{u}^{(0)}\nabla + 2\boldsymbol{r}\cdot\boldsymbol{u}_B^{(0)(i)}\nabla, \tag{17}$$

where $\boldsymbol{u}^{(0)}$ contains the sum of incident and scattered waves and $\boldsymbol{u}_B^{(0)(i)}$ is the zeroth-order incident shear wave of (14). The new complementary solution $\boldsymbol{u}_c^{(1)}$ thus now *consists entirely* of *diverging scattered waves* because the particular solution contains explicitly the first-order incident wave.

IV. Radiation condition in the perturbation method. In problems of elastic wave diffractions, the incident wave is specified and the scattered wave is to be determined which satisfies the appropriate boundary conditions at the obstacle and the Sommerfeld radiation condition at infinity. The total wave field is the sum of the incident and scattered waves. In the perturbation method, the nth-order incident wave is known and the nth-order scattered wave is to be determined, subject to the nth-order boundary conditions and an "nth-order radiation condition." The sum of the nth-order incident and scattered waves completes the nth-order solution.

To discuss radiation conditions we revert to the displacement potentials since these satisfy Helmholtz equations which are more familiar than the equations of motion (1) involving the displacements. With the displacement vector represented by

$$\boldsymbol{u} = \nabla\phi + \nabla\times\boldsymbol{h} \tag{18}$$

the equations of motion (1) are satisfied when the compressional and shear wave potentials, ϕ and $\boldsymbol{h}$ respectively, are solutions of

$$(\nabla^2 + k_1^2)\phi = 0; \qquad (\nabla^2 + k_2^2)\boldsymbol{h} = 0. \tag{19}$$

Denoting the scattered waves with superscripts (s), the Sommerfeld radiation condition for the above potentials becomes [6]:

$$\left.\begin{aligned} \phi^{(s)}(r) &\to \frac{F(\theta, \Psi)}{r}\exp(ik_1r) \\ \boldsymbol{h}^{(s)}(r) &\to \frac{\boldsymbol{G}(\theta, \Psi)}{r}\exp(ik_2r) \end{aligned}\right\} \quad \text{as} \quad r = |\boldsymbol{r}| \to \infty \tag{20}$$

where the functions F and $\boldsymbol{G}$ of the angular coordinates in a spherical polar coordinate system r, θ, Ψ are not specified explicitly.[1]

[1] The radiation condition for the displacement follows from Eqs. (20) and (18). They are discussed in [10], where the displacement is also separated into dilatational and rotational points.

We shall now present the analysis for the compressional waves, $\phi^{(s)}$, in detail. With $\phi^{(s)}$ satisfying the Sommerfeld condition (20), it may be shown [7] that at any point $\mathbf{r}$ in the field exterior to the obstacle

$$\phi^{(s)}(\mathbf{r}) = \iint_B \left[G(|\mathbf{R}|) \frac{\partial \phi^{(s)}(\boldsymbol{\varrho})}{\partial n} - \phi^{(s)}(\boldsymbol{\varrho}) \frac{\partial G(|\mathbf{R}|)}{\partial n} \right] dS \tag{21}$$

where B is the bounding surface of the obstacle, $\mathbf{n}$ is the unit outer normal to B, and $\boldsymbol{\varrho}$ denotes the position of the surface element dS. The Green's function G is given by

$$G(|\mathbf{R}|) = -\frac{1}{4\pi\,|\mathbf{R}|} \exp\,(ik_1\,|\mathbf{R}|) \tag{22}$$

with

$$|\mathbf{R}| = |\mathbf{r} - \boldsymbol{\varrho}| = (r^2 - 2\rho \cos\beta + \varrho^2)^{1/2} \tag{23}$$

and β is the angle between the "fixed" vector $\mathbf{r}$ and the "running" vector $\boldsymbol{\varrho}$. If we use spherical polar coordinates r, θ, Ψ for $\mathbf{r}$ and ρ, θ', Ψ' for $\boldsymbol{\varrho}$, it can be shown that

$$\cos\beta = \sin\Psi \sin\Psi' \cos(\theta - \theta') + \cos\Psi \cos\Psi'. \tag{24}$$

1. *Expansion of $\phi^{(s)}$ in power series in ϵ and r^{-1}.* To examine the scattered waves at a large distance from the obstacle, we expand G into a series in inverse powers of r:

$$G = -\frac{\exp\,(ik_1(r - \rho\cos\beta))}{4\pi r}\left[1 + \frac{\rho\cos\beta + k_1\rho^2(1 + \cos^2\beta)}{r} + \cdots\right], \tag{25a}$$

$$\begin{aligned}\frac{\partial G}{\partial n} = {} & \frac{\exp\,(ik_1(r - \rho\cos\beta))}{4\pi r} \\ & \cdot\left\{ik_1 \frac{\partial(\rho\cos\beta)}{\partial n} + \frac{1}{r}\left[ik_1 \frac{\partial(\rho\cos\beta)}{\partial n}(\rho\cos\beta + k_1\rho^2(1 + \cos^2\beta))\right.\right. \\ & \left.\left. - \frac{\partial}{\partial n}(\rho\cos\beta + k_1\rho^2(1 + \cos^2\beta))\right] + \cdots\right\}.\end{aligned} \tag{25b}$$

Since $\exp(-ik_1 r)G$ is analytic in $1/r$ for $r > \rho$, the series in inverse powers of r in the above equations are absolutely convergent for $r > \rho$. Moreover, by choosing r larger than the largest value of ρ along B, $\rho_{\max}$, the series converges uniformly over B. Thus, we may substitute Eqs. (25) into Eq. (21) and integrate term by term to obtain the uniformly and absolutely convergent series

$$\phi^{(s)}(\mathbf{r}; \epsilon) = \exp\,(ik_1 r)\left[\frac{f_1(\theta, \Psi; \epsilon)}{r} + \frac{f_2(\theta, \Psi; \epsilon)}{r^2} + \cdots\right] \tag{26}$$

where

$$f_1 = -\frac{1}{4\pi}\iint_B \exp\,(-ik_1\rho\cos\beta)\left[\frac{\partial\phi^{(s)}(\boldsymbol{\varrho}; \epsilon)}{\partial n} + ik_1 \frac{\partial(\rho\cos\beta)}{\partial n}\phi^{(s)}(\boldsymbol{\varrho}; \epsilon)\right] dS,$$

$$\begin{aligned}f_2 = {} & -\frac{1}{4\pi}\iint_B \exp\,(-ik_1\rho\cos\beta)\left\{\frac{\partial\phi^{(s)}}{\partial n}[\rho\cos\beta + k_1\rho^2(1 + \cos^2\beta)]\right. \\ & + \phi^{(s)}\left[ik_1 \frac{\partial(\rho\cos\beta)}{\partial n}(\rho\cos\beta + k_1\rho^2(1 + \cos^2\beta))\right. \\ & \left.\left. - \frac{\partial}{\partial n}(\rho\cos\beta + k_1\rho^2(1 + \cos^2\beta))\right]\right\} dS.\end{aligned}$$

Furthermore, if we assume that $\phi^{(s)}$ and $\partial\phi^{(s)}/\partial n$ are analytic functions of ϵ on B, then the functions $f_m(\theta, \Psi; \epsilon)$ will be analytic functions of ϵ since $k_1 = k(1 - 2\epsilon)^{1/2}$ is analytic in ϵ for $|\epsilon| < \frac{1}{2}$. Hence, we may expand each term in (26) in a series in ϵ and rearrange the terms to form a power series in ϵ. The final result is

$$\begin{aligned}\phi^{(s)}(r; \epsilon) = e^{ikr} \Bigg\{ &\left[\frac{f_1^{(0)}(\theta, \Psi)}{r} + \frac{f_2^{(0)}(\theta, \Psi)}{r^2} + \cdots\right] \\ &+ \epsilon\left[-ikf_1^{(0)}(\theta, \Psi) + \frac{1}{r}(f_1^{(1)}(\theta, \Psi) - ikf_2^{(0)}(\theta, \Psi)) + \cdots\right] \\ &+ \epsilon^2\left[-\frac{1}{2}k^2f_1^{(0)}(\theta, \Psi)r + \cdots\right] + \cdots\Bigg\}. \end{aligned} \tag{27}$$

In the derivation leading to Eq. (27) we have established that: (1) the ϵ-series expansion for $\phi^{(s)}$ is uniformly convergent outside B, and (2) the coefficients of ϵ^n in the ϵ-series, which is also the nth-order perturbation of the scattered wave, may be expanded into a uniformly and absolutely convergent series in $1/r$ for $r > \rho_{max}$. If (27) is expressed as

$$\phi^{(s)}(r; \epsilon) = \phi^{(s)(0)}(r) + \epsilon\phi^{(s)(1)}(r) + \cdots$$

it follows from (27) that

$$\phi^{(s)(n)}(r) = r^{n-1}e^{ikr}\sum_{m=0}^{\infty} F_m^{(n)}(\theta, \Psi)r^{-m}. \tag{28}$$

By applying a similar analysis to the Cartesian components of $\boldsymbol{h}$ in (19) and (20) with k_1 replaced by $k_2 = k(1 + 2\epsilon)^{1/2}$, we can deduce that

$$\boldsymbol{h}^{(s)}(r; \epsilon) = \boldsymbol{h}^{(s)(0)}(r) + \epsilon\boldsymbol{h}^{(s)(1)}(r) + \cdots$$

and

$$\boldsymbol{h}^{(s)(n)}(r) = r^{n-1}e^{ikr}\sum_{m=0}^{\infty} \boldsymbol{G}_m^{(n)}(\theta, \Psi)r^{-m}. \tag{29}$$

2. *Radiation condition for the nth order scattered waves.* For $r > \rho_{max}$ the series in (28) and (29) can be differentiated term by term to form the series for the nth-order displacements associated with the scattered waves according to Eq. (18). After rearranging the terms in a descending power series in r we find finally

$$\boldsymbol{u}^{(s)(n)}(r) = r^{n-1}e^{ikr}\sum_{m=0}^{\infty} \mathbf{U}_m^{(n)}(\theta, \Psi)/r^m. \tag{30}$$

Therefore, for r large the leading term for $\boldsymbol{u}^{(s)(n)}$ is given by

$$\boldsymbol{u}^{(s)(n)}(r) \to r^{n-1}\; U_0^{(n)}e^{ikr} \tag{31}$$

which we shall adopt as the "nth-order radiation condition" for the nth-order scattered wave.

In particular, the zeroth-order radiation condition is

$$\boldsymbol{u}^{(s)(0)}(r) \to U_0^{(0)}e^{ikr}/r. \tag{32}$$

This is precisely the Sommerfeld condition for the Cartesian components of $\boldsymbol{u}^{(s)(0)}$ which satisfy the homogeneous Helmholtz equation (5).

For the first-order displacements, the radiation condition becomes

$$u^{(s)(1)}(r) \to U_0^{(1)} e^{ikr}. \tag{33}$$

From (17) the first-order scattered wave is given by

$$u^{(s)(1)} = r \cdot \nabla u^{(s)(0)} - 2r \cdot u^{(s)(0)} \nabla + u_c^{(1)}. \tag{34}$$

In view of (32) the solution (34) thus satisfies the radiation condition (33), if the complementary solution $u_c^{(1)}$ meets the Sommerfeld radiation condition as in (20).

3. *Uniqueness theorem for the nth-order solution.* We now prove a uniqueness theorem for the complete scattered wave solution $u^{(s)(n)}$ which satisfies the radiation condition (31). Because it is the case of interest in the remaining two sections of this paper, we treat diffraction problems in which the displacements are prescribed at the boundary B of the obstacle.

The nth-order scattered wave which satisfies Eq. (3)

$$(\nabla^2 + k^2)u^{(n)(s)} = 2\nabla^2 u^{(n-1)(s)} - 4\nabla(\nabla \cdot u^{(n-1)(s)} + 4k^2 u^{(n-2)(s)}),$$

satisfies the boundary conditions $u^{(n)(s)} = u_B^{(n)}$ on B, the radiation condition (31), and has an expansion for large r in the form of (30), is unique. Note that one such solution exists, namely the coefficient of ϵ^n in the ϵ-expansion of the exact solution $u^{(s)}$. Also, in the theorem it is assumed that $u^{(n-1)(s)}$ and $u^{(n-2)(s)}$ have been determined uniquely.

Let there be two solutions to the nth-order problem and define their difference as v. Then

$$(\nabla^2 + k^2)v = 0; \qquad v = 0 \quad \text{on} \quad B.$$

Now, let $\mathcal{G}(r, r')$ be the solution of the following problem:

$$(\nabla^2 + k^2)\mathcal{G} = \delta(x - x', y - y', z - z'); \qquad \mathcal{G} = 0 \quad \text{on } B$$

and

$$\mathcal{G}(r, r') = \left[\frac{F_1(r, \theta', \Psi')}{r'} + \frac{F_2(r, \theta', \Psi')}{r'^2} + \cdots\right] e^{ikr'} \tag{35}$$

for r' sufficiently large. In other words, $\mathcal{G}$ is the scalar Green's function for the scattering problem which vanishes on B and satisfies the Sommerfeld radiation condition. The source and receiver points (r, r') are in the exterior field.

Application of Green's formula in the region bounded on the inside by B and outside by a large sphere S of radius R then yields

$$v(r) = \iint_S \left[v(R) \frac{\partial \mathcal{G}(r, R)}{\partial R} - \mathcal{G}(r, R) \frac{\partial v(R)}{\partial R}\right] dS \tag{36}$$

where the integral over B (not written) is identically zero because v and $\mathcal{G}$ vanish on B. In (37), r is a field point between B and S and, in spherical polar coordinates, $dS = R^2 \sin \Psi \, d\Psi \, d\theta$.

If we now let $R \to \infty$, we may substitute the expansion (35) for $\mathcal{G}$ in (36). Also, since the two solutions of the nth-order problem each have expansions in the form of (30) when $r > \rho_{\max}$, it follows that v has an expansion in the form

$$v(R) = R^{n-1} e^{ikR} \sum_{m=0}^{\infty} V_m(\theta, \Psi)/R^m \tag{37}$$

along the sphere S. Hence, Eq. (36) can be expressed as

$$v(r) = R^{n-1}e^{2ikR}\left[\hat{V}_0(r) + \frac{1}{R}\hat{V}_1(r) + \cdots + \frac{1}{R^{n-1}}V_{n-1}(r) + \frac{1}{R^n}\hat{V}_n(r) + O(R^{-n-1})\right] \tag{38}$$

where

$$\hat{V}_0(r) = -n\int_0^{2\pi}\int_0^{\pi} F_1(r, \theta, \Psi)V_0(\theta, \Psi)\sin\Psi\, d\Psi\, d\theta,$$

$$\hat{V}_1(r) = -\int_0^{2\pi}\int_0^{\pi}[(n-1)F_1V_1 + (n+1)F_2V_0]\sin\Psi\, d\Psi\, d\theta$$

and, in general, $\hat{V}_j$ is given by an integral over Ψ and θ whose integrand contains V_0, $V_1, \cdots, V_j$ in linear combination.

Hence, it is seen from (38) that as $R \to \infty$ the difference between the two solutions, each of which was assumed to exist, is either undefined at all points r (when not all $\hat{V}_j = 0$ for $j = 0, 1, \cdots, n-1$) or is zero (when all $\hat{V}_j = 0$ for $j = 0, 1, \cdots, n-1$). In this instance, the contradiction can be reconciled only by taking $\hat{V}_j = 0$ for $j = 0, 1, \cdots, n-1$ because we know at least one solution exists. Thus, our nth-order perturbation problem has a unique solution and it is the coefficient of ϵ^n in the ϵ-expansion of the exact solution.

To conclude this section we present the radiation conditions for two-dimensional scattering problems in the perturbation method. Here the obstacle is an infinitely long cylinder in an extended medium (plane strain) with a cross-sectional curve C. Following the previous steps in the three-dimensional analysis with the integral in (21) replaced by a line integral around C and using the two-dimensional Green's function for outgoing waves, we can show that

$$u^{(s)(n)}(r) \to r^{n-1/2}\mathrm{U}_0^{(n)}(\theta) \quad \text{as } r \to \infty. \tag{39}$$

Here $r = (x^2 + y^2)^{1/2}$ and r, θ are cylindrical polar coordinates. We note, again, that the first-order perturbation solution given by (17) satisfies the radiation condition above with $n = 1$.

In the next sections we study diffractions of plane elastic waves by: (1) a semi-infinite rigid-clamped strip, (2) a clamped strip of finite width. Zeroth-order solutions for both problems are the same as the corresponding ones for acoustic wave diffractions. The discussions, therefore, are centered mainly on the construction of the first-order solution.

V. Diffraction of elastic waves by a semi-infinite clamped strip. The problem described by the title of this section was investigated by Roseau [3], as discussed in Part I. Our own treatment of this problem in Part I contains an error in the first-order solution (Eq. (61)) due to an oversight on the plane wave perturbation. Using the method of perturbation of displacements, we derive the correct results in this section.

Let the xz half-plane, $x > 0$, represent a rigid, clamped barrier in an infinite elastic solid. Incident plane P and SV waves, represented by Eqs. (12) with

$$\mathbf{l} = (\cos\alpha, \sin\alpha, 0), \qquad \mathbf{m} = (\sin\beta, -\cos\beta, 0), \qquad \mathbf{n} = (\cos\beta, \sin\beta, 0), \tag{40}$$

impinge on this plane along which the boundary conditions are

$$\mathbf{u} = 0 \quad \text{at} \quad y = 0, \qquad x > 0. \tag{41}$$

As the problem is one of plane strain, we have

$$\mathbf{u} = [u(x, y), v(x, y), 0]$$

with the time factor $\exp(-i\omega t)$ omitted.

1. *Zeroth-order solution.* The zeroth-order incident waves follow from Eqs. (14) and (40):

$$\begin{aligned} u^{(0)(i)} &= A \cos\alpha \exp^{(ik\mathbf{r}\cdot\mathbf{l})} + B \sin\beta \exp^{(ik\mathbf{r}\cdot\mathbf{n})}, \\ v^{(0)(i)} &= A \sin\alpha \exp^{(ik\mathbf{r}\cdot\mathbf{l})} - B \cos\beta \exp^{(ik\mathbf{r}\cdot\mathbf{n})}. \end{aligned} \tag{42}$$

The boundary conditions are

$$u^{(0)} = v^{(0)} = 0 \quad \text{at} \quad y = 0, \qquad x > 0 \tag{43}$$

with $\mathbf{u}^{(0)} = \mathbf{u}^{(0)(i)} + \mathbf{u}^{(0)(s)}$.

Since the two components $u^{(0)}$ and $v^{(0)}$ are uncoupled both in the zeroth-order field equation (5) and the boundary conditions (43), the solution for each of them is the same as for the diffraction of acoustic waves by a "soft" screen (Dirichlet boundary condition). That solution was first given by Sommerfeld [8]. Following the notation in Part I, we obtain the zeroth-order displacement components as

$$\begin{aligned} u^{(0)} &= A \cos\alpha W_2(r, \alpha) + B \sin\beta W_2(r, \beta), \\ v^{(0)} &= A \sin\alpha W_2(r, \alpha) - B \sin\beta W_2(r, \beta) \end{aligned} \tag{44}$$

where, in polar coordinates r, θ, Sommerfeld's solutions for a plane incident wave at an angle α with the positive x-axis and unit amplitude, impinging on a "hard" or "soft" screen are, respectively,

$$\begin{aligned} W_{1,2}(r, \alpha) = \pi^{-1/2} \Big[&\exp(ikr\cos(\alpha - \theta)) \int_{a_-}^{\infty} e^{-t^2}\, dt \\ &\pm \exp(ikr\cos(\alpha + \theta)) \int_{a_+}^{\infty} e^{-t^2}\, dt \Big] \end{aligned} \tag{45}$$

with $a_{\pm} = (-2ikr)^{1/2} \sin\frac{1}{2}(\alpha \pm \theta)$. The above functions contain the sum of the incident and scattered waves.

The zeroth-order solution (44) satisfies the edge condition that the displacements remain bounded there, and it asymptotically yields the incident waves (42) as $r \to \infty$ outside the shadow region.

2. *First-order solution.* The particular solution of the first order (17) has the Cartesian components

$$\begin{aligned} u_p^{(1)} &= -x \frac{\partial u^{(0)}}{\partial x} + y \frac{\partial u^{(0)}}{\partial y} - 2y \frac{\partial v^{(0)}}{\partial x} + 2x \frac{\partial u_B^{(0)(i)}}{\partial x} + 2y \frac{\partial v_B^{(0)(i)}}{\partial x}, \\ v_p^{(1)} &= x \frac{\partial v^{(0)}}{\partial x} - y \frac{\partial v^{(0)}}{\partial y} - 2x \frac{\partial u^{(0)}}{\partial y} + 2x \frac{\partial u_B^{(0)(i)}}{\partial y} + 2y \frac{\partial v_B^{(0)(i)}}{\partial x}. \end{aligned} \tag{46}$$

Noting from (42) that

$$\frac{\partial u_B^{(0)(i)}}{\partial x} = -\frac{\partial v_B^{(0)(i)}}{\partial y}$$

and that $\partial/\partial\theta = x\partial/\partial y - y\partial/\partial x$, we rewrite (46) as

$$u_p^{(1)} = -x \frac{\partial u^{(0)}}{\partial x} + y \frac{\partial u^{(0)}}{\partial y} - 2y \frac{\partial v^{(0)}}{\partial x} - 2 \frac{\partial v_B^{(0)(i)}}{\partial \theta},$$

$$v_p^{(1)} = x \frac{\partial v^{(0)}}{\partial x} - y \frac{\partial v^{(0)}}{\partial y} - 2y \frac{\partial u^{(0)}}{\partial x} - 2 \frac{\partial}{\partial \theta} (u^{(0)} - u_B^{(0)(i)}).$$

The last term on the right-hand side of $v_p^{(1)}$ can be replaced by $-2\partial u_A^{(0)(i)}/\partial\theta$ because $u^{(0)} = u_A^{(0)(i)} + u_B^{(0)(i)} + u^{(0)(s)}$ and because $\partial u^{(0)(s)}/\partial\theta$ is a solution of the homogeneous Helmholtz equation which can be absorbed in the complementary solution for $v^{(1)}$. Hence, we finally arrive at a general first-order solution in the form

$$u^{(1)} = -x \frac{\partial u^{(0)}}{\partial x} + y \frac{\partial u^{(0)}}{\partial y} - 2y \frac{\partial v^{(0)}}{\partial x} - 2 \frac{\partial v_B^{(0)(i)}}{\partial \theta} + u_c^{(1)},$$
$$v^{(1)} = x \frac{\partial v^{(0)}}{\partial x} - y \frac{\partial v^{(0)}}{\partial y} - 2y \frac{\partial u^{(0)}}{\partial x} - 2 \frac{\partial u_A^{(0)(i)}}{\partial \theta} + v_c^{(1)}. \tag{47}$$

Although both θ-derivatives of the zeroth-order incident waves are complementary solutions, we shall not combine them with $u_c^{(1)}$ and $v_c^{(1)}$ because the latter must satisfy the radiation condition discussed in Sec. III.

Since $\mathbf{u}^{(0)} = \mathbf{u}^{(0)(i)} + \mathbf{u}^{(0)(s)}$, Eqs. (47) may be regarded as a superposition of the first-order incident wave $\mathbf{u}^{(1)(i)}$ and the first-order scattered wave $\mathbf{u}^{(1)(s)}$. The former is contained in the derivatives of $\mathbf{u}^{(0)(i)}$, which, according to Eq. (17), is the same as $\mathbf{u}^{(1)(i)}$ defined by Eqs. (15). The latter consists of the derivatives of $\mathbf{u}^{(0)(s)}$ and the complementary solution $\mathbf{u}_c^{(1)}$. The boundary condition for $\mathbf{u}^{(1)}$ is still the following:

$$\mathbf{u}^{(1)} = \mathbf{u}^{(1)(i)} + \mathbf{u}^{(1)(s)} = 0 \quad \text{on} \quad y = 0, \qquad x > 0. \tag{48}$$

However, since the first three terms in each of Eqs. (47) already satisfy the boundary condition (48), it is more convenient here not to separate explicitly the incident and scattered fields.

To find the complementary solutions, $u_c^{(1)}$ and $v_c^{(1)}$, we recall that $\partial W_1(r, \alpha)/\partial y = 0$ along the half-plane barrier. Since $\partial/\partial\theta = x\partial/\partial y$ on $y = 0$, we take

$$u_c^{(1)} = 2(\partial/\partial\theta)[B \cos \beta W_1(r, \beta) + v_B^{(0)(i)}],$$
$$v_c^{(1)} = 2(\partial/\partial\theta)[-A \cos \alpha W_1(r, \alpha) + u_A^{(0)(i)}]. \tag{49}$$

Note that the above expressions inside the brackets are the difference of a total wave field and an incident wave whose amplitudes are given by (42). Thus $\mathbf{u}_c^{(1)}$ contains the scattered wave perturbation alone. The final result for the first-order solution is

$$u^{(1)} = -x \frac{\partial u^{(0)}}{\partial x} + y \frac{\partial u^{(0)}}{\partial y} - 2y \frac{\partial v^{(0)}}{\partial x} + 2B \cos \beta \frac{\partial W_1(r, \beta)}{\partial \theta},$$
$$v^{(1)} = x \frac{\partial v^{(0)}}{\partial x} - y \frac{\partial v^{(0)}}{\partial y} - 2y \frac{\partial u^{(0)}}{\partial x} - 2A \cos \alpha \frac{\partial W_1(r, \alpha)}{\partial \theta}. \tag{50}$$

It can be verified that each term above vanishes at $y = 0$ for $x > 0$. Since the zeroth-order displacements are bounded at the edge $x = y = r = 0$ with

$$\partial W_{1,2}/\partial r = O(r^{-1/2}), \qquad \partial W_{1,2}/\partial\theta = O(r^{1/2}),$$

we have $\mathbf{u}^{(1)} = O(r^{1/2})$ as $r \to 0$, satisfying the edge condition.

3. *Stress intensification at the edge of the strip.* The stresses through the first order can be found from Hooke's Law (4) in which we take $\boldsymbol{u} \approx \boldsymbol{u}^{(0)} + \epsilon\boldsymbol{u}^{(1)}$. Near the edge, $r \to 0$, the leading terms in the stresses along the top and bottom edges of the strip ($y = 0$, $x > 0$) become[2]

$$\tau_{xx}(x, 0^{\pm}) = \frac{\nu}{1-\nu}\,\tau_{yy}(x, 0^{\pm}),$$

$$\tau_{yy}(x, 0^{\pm}) \approx \pm\tau_A \left(\frac{2}{\pi kx}\right)^{1/2} \left[\left(1 + \frac{\epsilon}{2}\right) \sin\alpha \cos\frac{\alpha}{2} - \epsilon \sin\frac{3\alpha}{2}\right] \exp\,(i(kx - 3\pi/4)), \tag{51}$$

$$\tau_{xy}(x, 0^{\pm}) \approx \pm\tau_A \left(\frac{2}{\pi kx}\right)^{1/2} \left(1 - \frac{3\epsilon}{2}\right) \cos\alpha \cos\frac{\alpha}{2} \exp\,(i(kx - 3\pi/4))$$

for an incident compressional wave alone ($B = 0$), and

$$\tau_{xx}(x, 0^{\pm}) = \frac{\nu}{1-\nu}\,\tau_{yy}(x, 0^{\pm}),$$

$$\tau_{yy}(x, 0^{\pm}) = \pm\tau_B \left(\frac{2}{\pi kx}\right)^{1/2} \left(1 + \frac{3\epsilon}{2}\right) \cos\beta \cos\frac{\beta}{2} \exp\,(i(kx - 3\pi/4)), \tag{52}$$

$$\tau_{xy}(x, 0^{\pm}) = \pm\tau_B \left(\frac{2}{\pi kx}\right)^{1/2} \left[\left(1 - \frac{\epsilon}{2}\right) \sin\beta \cos\frac{\beta}{2} + \epsilon \sin\frac{3\beta}{2}\right] \exp\,(i(kx - 3\pi/4))$$

for an incident shear wave alone ($A = 0$). In the above

$$\tau_A = i\mu k_2^2 A/k \quad \text{and} \quad \tau_B = -i\mu k_2^2 B/k$$

are the magnitudes of normal and shear stress of the incident compressional and shear waves, respectively.

VI. Diffraction of elastic waves by a finite clamped strip. Consider a rigid strip of width 2a clamped symmetrically along the x-axis. Its edges are then at $y = 0$, $x = \pm a$. To shorten the calculation, but still not miss the essential feature which we want to convey—edge behavior of the first-order particular solution—we take a plane compressional wave propagating perpendicular to the strip. From (12) the incident wave becomes ($B = 0$, $\alpha = \pi/2$)

$$u^{(i)} = 0, \qquad v^{(i)} = A \exp\,(ik_1 y). \tag{53}$$

The boundary conditions are

$$\boldsymbol{u} = \boldsymbol{u}^{(i)} + \boldsymbol{u}^{(s)} = 0 \quad \text{at} \quad y = 0, \quad |x| < a, \tag{54}$$

and $\boldsymbol{u}$ remains finite at the edges $x = \pm a$, $y = 0$.

1. *Zeroth-order solution.* The perturbation method will be applied to determine the displacement field near the rigid strip and at low frequencies. As in the previous section, the zeroth-order solution is the same as for scattering of sound waves by a soft screen [9]. In elliptic coordinates (ξ, η) with the transformation

$$x = a \cosh\xi \cos\eta, \qquad y = a \sinh\xi \sin\eta, \qquad 0 < \eta < 2\pi,$$

[2] Eqs. (50), (51), and (52) replace Eqs. (61), (63), and (64) of Part I, respectively.

the solution can be expressed in terms of the Mathieu functions. For a plane wave of amplitude A and angle $\pi/2$, we obtain

$$u^{(0)} = 0,$$
$$v^{(0)} = Ae^{iky} - 2A \sum_{n=0}^{\infty} B_n(q) Mc_{2n}^{(3)}(\xi, q) ce_{2n}(\eta, q), \tag{55}$$

where

$$B_n(q) = (-1)^n ce_{2n}(\pi/2, q) Mc_{2n}^{(1)}(0, q)/Mc_{2n}^{(3)}(0, q),$$
$$q = k^2a^2/4$$

and

$$ce_{2n}(\eta, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)}(q) \cos 2r\eta,$$

$$Mc_{2n}^{(j)}(\xi, q) = [ce_{2n}(0, q)]^{-1} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)}(q) Z_{2r}^{(j)}(2\sqrt{q} \cosh \xi)$$

are the Mathieu function and modified Mathieu functions of the jth kind, respectively.[3] $Z_n^{(j)}(z)$ represents a circular cylinder function with $Z_n^{(1)}(z) = J_n(z)$ and $Z_n^{(3)}(z) = H_n^{(1)}(z)$. The coefficients $A_{2r}^{(2m)}(q)$ are known and tabulated for small values of q.

2. *First-order solution.* The general solution, Eq. (47), of the first-order equation derived in the previous section is applicable for all scattering problems involving plane incident waves. For the case of a normally incident compressional wave, substitution of (55) into (47) yields

$$u^{(1)} = -2y\, \partial v^{(0)}/\partial x + u_c^{(1)},$$
$$v^{(1)} = x\, \partial v^{(0)}/\partial x - y\, \partial v^{(0)}/\partial y + v_c^{(1)}. \tag{56}$$

It remains to determine outgoing wave complementary solutions, $u_c^{(1)}$ and $v_c^{(1)}$, so as to satisfy the boundary conditions

$$u^{(1)} = v^{(1)} = 0 \quad \text{at} \quad y = 0, \quad |x| < a, \tag{57}$$

and the edge conditions at $x = \pm a$, $y = 0$.

Since $2y\, \partial v^{(0)}/\partial x$ vanishes on $y = 0$ and remains finite at $x = \pm a$ as $y \to 0$, we have simply that $u_c^{(1)} = 0$. The first two terms of $v^{(1)}$ also vanish on $y = 0$, $|x| < a$, but $x\, \partial v^{(0)}/\partial x$ will be singular at the edges as shown below. Thus $v_c^{(1)}$ must be chosen to satisfy the boundary condition (57) and to eliminate singularities at the edges.

We proceed by introducing polar coordinates, locally at each edge, with (ρ_+, γ_+) originating at $x = a$, $y = 0$ ($\xi = \eta = 0$) and (ρ_-, γ_-) originating at $x = -a$, $y = 0$ ($\xi = 0$, $\eta = \pi$). Thus, at the two edges, we have

$$x = \pm a + \rho_\pm \cos \gamma_\pm, \qquad y = \rho_\pm \sin \gamma_\pm \tag{58}$$

where

$$-\pi < \gamma_+ < \pi, \qquad 0 < \gamma_- < 2\pi.$$

[3] The notation for the modified Mathieu functions used here follows that of the National Bureau of Standards, *Handbook of mathematical functions*, edited by M. Abramowitz and I. A. Stegun, Dover (1965).

Near the edges, the elliptic coordinates are transformed to the local polar coordinates as

$$\xi \to (2\rho_\pm/a)^{1/2} \begin{cases} \cos(\gamma_+/2) \\ \sin(\gamma_-/2), \end{cases} \qquad \eta \to \begin{cases} (2\rho_+/a)^{1/2} \sin \gamma_+/2 \\ \pi - (2\rho_-/a)^{1/2} \cos(\gamma_-/2). \end{cases}$$

Carrying out the differentiation of $v^{(0)}$ in (56) according to

$$x \frac{\partial}{\partial x} = a \cosh \xi \cos \eta \left[\frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} \right]$$

and noting that $ce'_{2n}(0) = ce'_{2n}(\pi) = 0$ and $ce_{2n}(0) = ce_{2n}(\pi)$, we find, as $\xi \to 0$ and $\eta \to 0$ or π, $(\rho_\pm \to 0)$ that

$$x\, \partial v^{(0)}/\partial x \to -C(a/2\rho_\pm)^{1/2} \begin{cases} \cos(\gamma_+/2) \\ \sin(\gamma_-/2) \end{cases} \tag{59}$$

where

$$C = 2A \sum_{n=0}^{\infty} B_n(q) Mc_{2n}^{(3)}(0, q) ce_{2n}(0, q)$$

which is a complex coefficient depending on q. In the above derivation use has been made of the relation

$$x \frac{\partial \xi}{\partial x} = \frac{\sinh 2\xi \cos^2 \eta}{2(\sinh^2 \xi + \sin^2 \eta)} \to \left(\frac{a}{2\rho_\pm}\right)^{1/2} \begin{cases} \cos(\gamma_+/2) \\ \sin(\gamma_-/2) \end{cases}$$

as $\rho_\pm \to 0$ $(\xi \to 0, \eta \to 0, \pi)$.

Eq. (59) shows that $v^{(1)}$ indeed contains a singular term at each edge as $\rho_\pm \to 0$ which must be cancelled by a corresponding term in the complementary solution. Therefore, we choose

$$v_c^{(1)} = C[(a/2\rho_+)^{1/2} \exp(ik\rho_+) \cos(\gamma_+/2) + (a/2\rho_-)^{1/2} \exp(ik\rho_-) \sin(\gamma_-/2)]. \tag{60}$$

Each term in brackets in (60) represents a diverging wave from an edge of the strip, satisfying the Helmholtz equation and vanishing identically on the strip ($\gamma_+ = \pm\pi$, $\gamma_- = 0, 2\pi$). Substitution of (60) and $u_c^{(1)} = 0$ into (56) completes the first-order solution.

3. *Discussion.* The perturbation solution of zeroth- and first-orders derived above for diffraction of a P wave by a finite rigid strip serves as an approximation to the solution of this problem. So far no exact results for the entire field have been explicitly obtained. As mentioned in the introduction, Ang and Knopoff [4] and Harumi [5] have also treated this problem, the former using an integral equation formulation, the latter using the wave function expansion method in elliptic coordinates. In both works, approximate solutions were derived which are useful for calculations in the field away from the strip at low frequencies. Here, each of the perturbation solutions presented above is exact and, as explained in Part I, can be expected to provide reliable accuracy for the field near the strip at low frequencies, including results for the stress-intensity factors at its edges.

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