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SLOW MOTION OF A VISCOUS CONDUCTING FLUID PAST A SPHERE IN THE PRESENCE OF A TOROIDAL MAGNETIC FIELD*

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1. Introduction. One of the problems of steady flow magnetohydrodynamics which has attracted considerable attention during the last decade is the determination of the flow past a fixed obstacle in the presence of a magnetic field. The early papers dealt with the situation in which the free magnetic field is uniform and aligned with the uniform stream at infinity. More recently, several authors have been concerned with the case in which the free field originates inside the obstacle. In particular Barthel and Lykoudis [1] have considered the problem of flow past a fixed magnetized sphere in which there is an axially symmetric magnetic dipole located at the centre. The velocity field was calculated for small values of the Hartmann number and an expression found for the drag coefficient. Tamada and Sone [2] have obtained similar results, and in addition have found an asymptotic solution for large values of the Hartmann number. Riley [3] has considered the case in which the sphere is magnetized by a magnetic pole at its centre. In this problem the flow equations are reduced to a pair of coupled ordinary differential equations and a solution can be found in series form for arbitrary Hartmann number.

In this paper the field originates inside the sphere which is supposed conducting, and is produced by an electric current dipole located at the centre of the sphere and directed along the axis of symmetry. However, the associated magnetic field inside the sphere is toroidal, consisting of one component perpendicular to the meridional plane. It is shown that both the hydrodynamic and electromagnetic boundary conditions can be satisfied provided that the fluid velocity is axially symmetric (without swirl) and the magnetic field in the fluid is toroidal. The Stokes flow perturbations are calculated for small values of the Reynolds number R, the magnetic Reynolds number R_m and the Hartmann number M. The primary interest is in the effect of the magnetic field on the fluid motion near the sphere and the vorticity generation at the boundary.

For $M^2 > 6R_m$ a region of reversed flow occurs about the forward stagnation point and spreads over the front face of the sphere with increasing values of M^2/R_m. The force on the sphere is found to be reduced by a factor of $O(M^2)$ of the Stokes drag, from which it is concluded the effect of the term of order M^2 is to retard the fluid motion, consequently reducing the viscous stresses around the sphere and hence the drag. The magnetic field also contributes to a force on the sphere which also reduces the drag by a term of order M^2. It is shown that the dominant characteristics of the flow field are not

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affected by the magnetic field at large distances from the sphere, and the magnetic field decays exponentially upstream and algebraically downstream.

Finally, it may be anticipated that the results presented here are typical of toroidal fields produced by electric current multipoles at the origin or a linear combination of such singularities.

Flow equations. The conventional magnetohydrodynamic equations describing the steady flow of a viscous conducting fluid are

$$\begin{gathered} \frac{\mu}{\rho_0}[\vec{H} \wedge \operatorname{curl} \vec{H}] + (\vec{q}\cdot\vec{\nabla})\vec{q} = -\operatorname{grad}\frac{p}{\rho_0} + \nu\nabla^2 q \\ \vec{j} = \operatorname{curl}\vec{H} = \sigma\vec{E} + \sigma\mu[\vec{q} \wedge \vec{H}] \\ \operatorname{curl}\vec{E} = 0, \qquad \operatorname{div}\vec{q} = 0, \qquad \operatorname{div}\vec{H} = 0, \end{gathered} \tag{1}$$

where $\vec{q}$ is the fluid velocity, $\vec{H}$ the magnetic field, $\vec{E}$ the electric field, p the pressure, ρ_0 the density, μ the permeability, $\vec{j}$ the current density, and σ the conductivity. Consider an axially symmetric fluid motion without swirl in the presence of a toroidal magnetic field. Then $\vec{q}$ and $\vec{H}$ may be expressed in the forms

$$\vec{q} = \operatorname{curl}\left\{\frac{-\psi}{\omega}\hat{\phi}\right\}, \qquad \vec{H} = \frac{U}{\omega}\hat{\phi}, \tag{2}$$

where (z, ω, ϕ) are cylindrical polar coordinates and $\hat{\phi}$ is the unit vector directed perpendicular to the azimuthal plane ϕ = constant and in the sense of ϕ increasing. $\psi = \psi\ (z, \omega)$ is the Stokes stream function and $U = U(z, \omega)$ may be regarded as a flux function for the current density $\vec{j}$. In terms of ψ and U Eqs. (1) simplify to

$$\frac{2\mu}{\rho_0}\frac{U}{\omega^2}\frac{\partial U}{\partial z} + \frac{\omega\partial\{\psi, L_{-1}(\psi)/\omega^2\}}{\partial(z, \omega)} = \nu L_{-1}^2(\psi), \tag{3}$$

$$\sigma\mu\frac{\omega\partial\{\psi, U/\omega^2\}}{\partial(z, \omega)} = L_{-1}(U), \tag{4}$$

where the Stokes operator

$$L_{-1} \equiv \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial\omega^2} - \frac{1}{\omega}\frac{\partial}{\partial\omega}.$$

To determine how flows of the type defined by (2) can arise, let S be a fixed conducting solid of revolution of permeability μ and conductivity σ. Then the electric and magnetic fields inside S are governed by the equations

$$\vec{j}_i = \operatorname{curl}\vec{H}_i = \sigma\vec{E}_i\,, \qquad \operatorname{curl}\vec{E}_i = 0, \qquad \operatorname{div}\vec{H}_i = 0 \tag{5}$$

where the subscript i denotes the fields inside S. Now if the magnetic field is toroidal inside S then

$$\vec{H}_i = \frac{U_i}{\omega}\hat{\phi}, \qquad L_{-1}(U_i) = 0. \tag{6}$$

In this U_i may be regarded as a flux function for both the current density and electric field. Now consider flow past S due to a uniform stream of velocity $-V_0\hat{k}$ at infinity directed along the negative z-axis in the presence of a magnetic field originating inside S and of the type described by (6). The boundary conditions for the fluid velocity are

$$\frac{1}{\omega}\frac{\partial\psi}{\partial n} = 0 = \frac{1}{\omega}\frac{\partial\psi}{\partial S} \quad \text{on} \quad C, \qquad \psi \sim \tfrac{1}{2}V_0\omega^2 \quad \text{as} \quad r = (\omega^2 + z^2)^{1/2} \to \infty \tag{7}$$

where $\partial/\partial n$, $\partial/\partial S$ denote differentiation along the outward drawn normal and along the tangent to the meridian curve C bounding S respectively. The electromagnetic boundary conditions are

(i) the magnetic field is continuous on C, that is

$$U = U_i \quad \text{on} \quad C \tag{8}$$

(ii) the tangential component of the electric field is continuous on C, that is

$$\hat{n} \wedge \vec{E} = \hat{n} \wedge \vec{E}_i \quad \text{on} \quad C. \tag{9}$$

In terms of U and U_i (9) may be expressed as

$$\frac{\partial U}{\partial n} = \frac{\partial U_i}{\partial n} \quad \text{on} \quad C. \tag{10}$$

In addition the field vanishes at infinity so that

$$|\vec{\mathrm{H}}| = |U/\omega| \to 0 \quad \text{as} \quad r \to \infty.$$

Exterior to S, ψ and U are governed by (3) and (4).

Flow past a sphere. Now let S be a sphere of radius a and centered at the origin of coordinates and suppose that the field inside S is produced by an electric or current dipole directed along the axis of symmetry.[1] Thus if $z = r\cos\theta$, $\omega = r\sin\theta$ define spherical polar coordinates then

$$U_i \sim M_0 \sin^2\theta/r \quad \text{as}\ r \to 0 \tag{11}$$

where M_0 is the dipole strength. If nondimensional quantities as defined by

$$\vec{r} = a\,\vec{r}^{\,1}, \qquad \vec{q} = V_0\vec{q}^{\,1}, \qquad \psi = V_0 a^2\psi^1, \qquad U = M_0U^1/a,$$
$$R = \frac{V_0 a}{\nu}, \qquad R_m = \frac{V_0 a}{\eta}, \qquad M^2 = \frac{\mu}{\rho_0}\frac{M_0^2}{a}\frac{1}{\eta\nu}, \qquad \vec{E} = \frac{M_0}{a^3\sigma}\vec{E}^1, \tag{12}$$

are substituted into (3) and (4), and then primes are removed, the scaled equations of flow are

$$\frac{2M^2}{R_m\omega^2}U\frac{\partial U}{\partial z} + R\omega\frac{\partial\{\psi, L_{-1}(\psi)/\omega^2\}}{\partial(z, \omega)} = L_{-1}^2(\psi), \tag{13}$$

$$R_m\omega\frac{\partial\{\psi, U/\omega^2\}}{\partial(z, \omega)} = L_{-1}(U), \tag{14}$$

with boundary conditions

$$\psi = \frac{\partial\psi}{\partial r} = 0, \qquad r = 1, \qquad \psi \sim \tfrac{1}{2}r^2(1 - \beta^2) \quad \text{as} \quad r \to \infty,$$
$$\frac{U}{\omega} \to 0 \quad \text{as} \quad r \to \infty, \qquad U = U_i, \qquad \frac{\partial U}{\partial r} = \frac{\partial U_i}{\partial r}\ r = 1, \qquad \beta = \cos\theta, \tag{15}$$

and $U_i \sim \sin^2\theta/r$ as $r \to 0$. R is the Reynolds number, R_m the magnetic Reynolds number and M the Hartmann number.

[1] The results do not depend on whether or not the dipole is directed along the $\pm$ z-axis.

Slow motion solution. In view of the coupled nonlinearity of (13) and (14) it is not possible to determine an exact analytical solution even for special values of the hydromagnetic parameters. For small values of the parameters it is customary to employ perturbation techniques to determine approximations to the fields. Now since the only singularity of the magnetic field is located at the centre of the sphere and the field vanishes at infinity, the primary interest is to determine the flow in a vicinity of the spherical boundary where the dominant magnetic field effects are present. Thus, if all three hydromagnetic parameters are small, the convection terms in the equations of motion will be neglected since Reynolds number effects are only of secondary interest and the first order effects are $O(R)$. It is readily seen from (13) and (14) that there are expansions[2] for ψ, U, U_i of the forms

$$\psi = \psi_0^0 + M^2R_m^{-1}\psi_{-1}^1 + M^2\psi_0^1 + M^4R_m^{-1}\psi_{-1}^2 + \cdots , \tag{16}$$

$$U = U_0^0 + R_m U_1^0 + M^2 U_0^1 + \cdots , \tag{17}$$

$$U_i + U_{i0}^0 + R_m U_{i1}^0 + M^2 U_{i0}^1 + \cdots , \tag{18}$$

where the superscript indicates the power of M^2 while the subscript denotes the power of R_m. Now in order to preserve consistency in respect to the assumptions already made it is clear that convection terms like

$$\frac{RM^4}{R_m^2}\omega \frac{\partial\{\psi_{-1}^1, L_{-1}(\psi_{-1}^1)/\omega^2\}}{\partial(z, \omega)}$$

are small in a vicinity of the sphere provided that $R^{1/2}M^2R_m^{-1} \ll 1$. This condition also ensures that higher order terms also contribute convections effects which are small in a vicinity of the boundary. Now for a conducting fluid such as mercury $R_m/R = O(10^{-7})$ so that if $R^{1/2}M^2R_m^{-1} = O(10^{-2})$, say, then for $R = O(10^{-6})$, $R_m = O(10^{-13})$, and $M^2R_m^{-1} = O(10)$, $M = O(10^{-6})$. It will be shown later that the case $M^2R_m^{-1} = O(10)$ is the flow of most interest in the present paper. Substitution of (16) and (17) in (13) and (14) and application of the boundary conditions yields the leading terms which are found to be

$$U_0^0 = (1 - \beta^2)/r, \qquad U_{i0}^0 = (1 - \beta^2)/r, \qquad \beta = \cos\theta, \tag{19}$$

$$\psi_0^0 = (\tfrac{1}{2}r^2 - \tfrac{3}{4}r + 1/4r)(1 - \beta^2), \tag{20}$$

where ψ_0^0 is the well-known Stokes flow for the field free case. The determination of ψ_{-1}^1 is then straightforward and is given by

$$\psi_{-1}^1 = \tfrac{1}{4}(1/r - 1/2r^2 - \tfrac{1}{2})\beta(1 - \beta^2). \tag{21}$$

Now the equation satisfied by U_1^0 is

$$L_{-1}(U_1^0) = \frac{-3(1-\beta^2)}{r^4}\frac{\partial\psi_0^0}{\partial\beta}, \tag{22}$$

subject to the boundary conditions

$$U_1^0 \to 0 \quad \text{as} \quad r \to \infty, \qquad U_1^0 = U_{i1}^0, \qquad \frac{\partial U_1^0}{\partial r} = \frac{\partial U_{i1}^0}{\partial r}, \qquad r = 1, \tag{23}$$

[2] The complete expansions are of the forms $\psi = \sum_{p=-1}^{\infty}\sum_{q=0}^{\infty} R_m^p M^{2q}\psi_p^q$, $U = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} R_m^p M^{2q} U_p^q$, etc.

where U_{i1}^0 is a solution of $L_{-1}(U_{i1}^0) = 0$, which is finite at the origin (or at least less singular than U_{i0}^0 at the origin). The solutions are found to be

$$U_{i1}^0 = -\tfrac{1}{40}r^3\beta(1 - \beta^2) \tag{24}$$

$$U_1^0 = (-\tfrac{1}{2} + 9/8r + 1/4r^3 - 9/10r^2)\beta(1 - \beta^2). \tag{25}$$

The term of order M^2, that is U_0^1, can be found in a similar manner. For the purposes of calculating the drag coefficient the term ψ_0^1 is required. This function satisfies the equation

$$L_{-1}^2(\psi_0^1) = \frac{2}{r^2(1 - \beta^2)}\left\{U_0^0\frac{\partial U_1^0}{\partial z} + U_1^0\frac{\partial U_0^0}{\partial z}\right\}, \tag{26}$$

subject to the boundary conditions $\psi_0^1 = \partial\psi_0^1/\partial r = 0$, $r = 1$, and $\psi_0^1 \to 0$ as $r \to \infty$. After some calculation the solution is found to be of the form

$$\psi_0^1 = F(r)(1 - \beta^2) + G(r)(5\beta^2 - 1)(1 - \beta^2), \tag{27}$$

where F and G are given by

$$F(r) = \frac{2}{5}\left\{\frac{3\log r}{40r} - \frac{1}{16} - \frac{1}{280r^3} + \frac{3}{80r^2} + \frac{11}{560r} + \frac{1}{112}r\right\}, \tag{28}$$

$$G(r) = \frac{2}{5}\left\{\frac{9}{80r}\log r + \frac{1}{12} + \frac{9\log r}{224r^3} - \frac{3}{20r^2} + \frac{401}{6720r^3} + \frac{109}{2240r}\right\}. \tag{29}$$

The most interesting case occurs when $M^2R_m^{-1} > 6$, and the stream function may be represented in the form

$$\psi = \frac{(r-1)^2}{4r}\left\{2r + 1 - \frac{3\cos\theta}{r\cos\theta_0}\right\}\sin^2\theta, \tag{30}$$

where $M^2/6R_m = \alpha/6 = \sec\theta_0$. It follows that the sphere and axis are streamlines and also the curve C defined by

$$2r + 1 - 3\cos\theta/\, r\cos\theta_0 = 0. \tag{31}$$

This curve C encloses a vortex or a region of reversed flow exterior to the forward portion of the sphere and meets the sphere at the point $P_0(1, \theta_0)$ and the axis at the point

$$r = \tfrac{1}{4}(1 + 4\alpha)^{1/2} - \tfrac{1}{4}, \qquad \theta = 0. \tag{32}$$

The fluid velocity along the axis is positive for $1 < r < \frac{1}{4}(1 + 4)^{1/2} - \frac{1}{4}$, $\theta = 0$, and this latter point is a stagnation point for the flow. There is also a stagnation point inside the region bounded by C and exterior to the sphere defined by the equations

$$4r(2r + 1)\beta + \alpha(1 - 3\beta^2) = 0, \qquad r(4r^2 + r + 1) - \alpha\beta = 0. \tag{33}$$

Now let λ be the angle between the tangent and the radius vector and mesured counter-clockwise from the radius; then

$$\tan\lambda = -2r(4r + 1)/\alpha\sin\theta. \tag{34}$$

Thus as $\theta \to 0$, $\tan\lambda$ becomes infinite so that the closed streamline intersects the axis orthogonally. Again for $0 < \theta \leq \pi/2$, $\tan\lambda < 0$, indicating that λ is obtuse. When $r = 1$, $\tan\lambda = -10(\alpha^2 - 36)^{-1/2}$, implying that the curve C comes in below the radius vector. Now the sign of ψ depends on the last factor contained in (30). If local polar coordinates (ρ, ϕ) are defined by

$$r \sin \theta = \sin \theta_0 + \rho \sin (\theta_0 + \phi), \qquad r \cos \theta = \cos \theta_0 + \rho \cos (\theta_0 + \phi) \tag{35}$$

then applying the approximation $(1 + 2\rho \cos \phi + \rho^2)^{1/2} = 1 + \rho \cos \phi$ for small ρ, the sign of ψ locally at the point of separation depends on

$$\rho(5 \cos \theta_0 \cos \phi + 3 \sin \theta_0 \sin \phi) \tag{36}$$

where ϕ is measured counterclockwise from the outer normal to the sphere. Since $0 \leq \theta_0 \leq \pi/2$ it is clear that $\psi > 0$ for $\theta \leq \phi \leq \pi/2$ and $\psi = 0$ for

$$\tan \phi = -\tfrac{5}{3} \cot \theta_0 = -10(\alpha^2 - 36)^{-1/2}. \tag{37}$$

Since $-\pi/2 \leq \phi \leq \pi/2$ the angle A defined by (37) is negative and

$$\psi > 0, \qquad A < \phi \leq \pi/2, \tag{38}$$

$$\psi < 0, \qquad -\pi/2 \leq \phi \leq A < 0. \tag{39}$$

The vorticity of the fluid arising from the dominant terms is given by

$$\zeta = \frac{1}{r \sin \theta} L_{-1}(\psi) = \frac{1}{r \sin \theta} \left\{ L_{-1}(\psi_0^0) + \frac{M^2}{R_m} L_{-1}(\psi_{-1}^1) \right\} \tag{40}$$

$$= \left\{ \frac{3}{2r^2} + \frac{n^2}{4R_m} \left(\frac{3}{r^3} - \frac{4}{r^4} \right) \cos \theta \right\} \sin \theta, \tag{41}$$

and its value on the boundary $r = 1$ is

$$\zeta = \frac{3}{2} \left(1 - \frac{M^2}{6R_m} \cos \theta \right) \sin \theta. \tag{42}$$

Thus, if M and R_m are small with $M^2 R_m^{-1}$ small then the vorticity is positive $0 < \theta < \pi$ and is decreased on the front face of the sphere $0 < \theta < \pi/2$ and increased on the rear face $\pi/2 < \theta < \pi$. Now if Reynolds number terms are included in the expansion then it is found the vorticity is increased by a term of $O(R)$ on the front face and decreased on the rear. The magnetic field thus inhibits Reynolds number effects in a vicinity of the sphere. For the more interesting situation in which $M^2/R_m > 6$, the vorticity vanishes at $\theta = \theta_0$ and is in fact negative for $0 < \theta < \theta_0$. This is of course indicative of a region of reversed flow about the forward stagnation point.

Finally it is noted that the z-component

$$\frac{M^2}{R_m} \frac{U}{\omega} \frac{\partial U}{\partial z} = \frac{3M^2 \sin^3 \theta \cos \theta}{r^4} \tag{43}$$

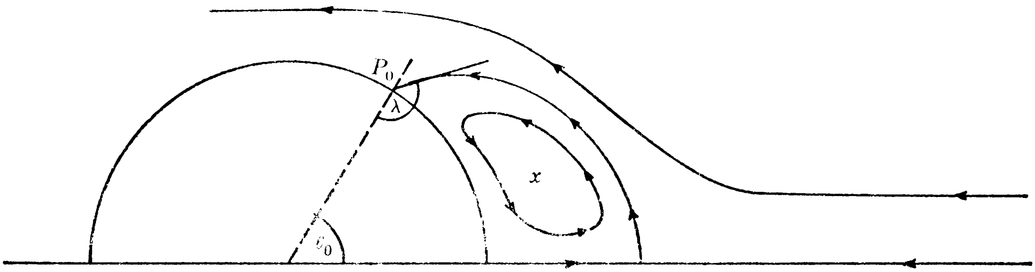

FIG. 1. Sketch of the streamlines for $M^2 > 6R_M$ illustrating region of reversed flow about forward stagnation point

of the Lorentz force which produces the vortex corresponds to a symmetric push and pull in the fluid medium and is clearly directed against the flow for $0 < \theta < \pi/2$ and with the fluid for $\pi/2 < \theta < \pi$.

Magnetic field at large distance from the sphere. At large distances from the sphere $\psi \sim \frac{1}{2}\omega^2$ so that the governing equation (14) for U may be obtained by replacing ψ for $\frac{1}{2}\omega^2$, and U satisfies an Oseen type equation given by

$$(L_{-1} + R_m^{\bullet}\, \partial/\partial z)U = 0. \tag{44}$$

Now as $r \to \infty$, $U/\omega \to 0$, and the inner boundary conditions are $U = U_i$, $\partial U/\partial r = \partial U_i/\partial r$, $r = 1$ and $U_i \sim \sin^2 \theta/r$ as $r \to 0$. For small values of R_m a solution of (44) satisfying the outer boundary and the inner boundary conditions correct to zero order in R_m is

$$U = \exp \frac{-R_m r(1 + \beta)}{2} \left[\frac{1}{r} + \frac{R_m}{2}\right] (1 - \beta^2), \tag{45}$$

$$U_i = \frac{(1 - \beta^2)}{r}. \tag{46}$$

The significance of this solution is that it shows that at upstream infinity ($\beta = 1$) U decays exponentially while at downstream infinity ($\beta = -1$), U decays algebraically in the wake.

Velocity field at large distances. At large distances from the sphere the governing equation for the dominant terms, $\psi = \psi_0^0 + M^2\psi_{-1}^1/R_m$ is the inhomogeneous Oseen equation given by

$$(L_{-1} + R\, \partial/\partial z)L_{-1}(\psi) = -6M^2\beta(1 - \beta^2)/R_m r^5. \tag{47}$$

Now ψ_0^0 is a solution of the homogeneous equation and the approximate solution is

$$\psi_0^0 = \tfrac{1}{2}r^2(1 - \beta^2) + (1 - \beta^2)/4r - (3/2R)(1 - \beta)[1 - \exp(-Rr(1 + \beta)/2), \tag{48}$$

and from (21) it is readily shown that the dominant term in ψ_{-1}^1 can be found from the solution of the homogeneous Oseen equation given by

$$\frac{M^2}{4RR_m}\left(\frac{R}{2} + \frac{1}{r}\right)(1 - \beta^2) \exp(-Rr(1 + \beta)/2) - \frac{M^2}{4RR_m}\frac{(1 - \beta^2)}{r}, \tag{49}$$

which behaves as $(M^2/R_m)(1 - \beta^2)\beta/8$ for small R and finite r ($\beta \neq -1$). It is readily found from (48) and (49) that the vorticity decays exponentially upstream ($\beta = 1$) and decays algebraically downstream ($\beta = -1$) at the same rate as the ordinary field free hydrodynamic case. Thus the magnetic field does not affect the dominant characteristics of the flow field at infinity.

The force on the sphere. The force on the sphere is composed of two parts, namely, the force due to the dynamic viscous stresses in the fluid and a force derived from the magnetic field stresses. The former contribution is expressed by a formula due to Stimson and Jeffery [4], that is,

$$\begin{aligned} F_D &= \pi\rho_0\nu a V_0 \int_C \omega^3 \frac{\partial}{\partial n}\left(\frac{L_{-1}(\psi)}{\omega^2}\right) ds \\ &= -6\pi\rho_0\nu a V_0\left\{1 - \frac{8M^2}{3150}\right\}. \end{aligned} \tag{50}$$

The latter contribution is given by

$$F_m = 2\pi\rho_0\nu a V_0 \frac{M^2}{R_m} \int_C [(\vec{H}\cdot\hat{k})(\vec{H}\cdot\hat{r}) - \tfrac{1}{2}\vec{H}^2(\hat{r}\cdot\hat{k})]\omega\, ds$$
$$= \frac{1}{150}\pi\rho_0\nu a V_0 M^2, \tag{51}$$

and the total force on the sphere is

$$F = F_D + F_M = -6\pi\rho_0\nu a V_0(1 - 23M^2/6300), \tag{52}$$

the negative sign indicating that F is directed along the negative z-axis. It is noted that terms like M^2/R_m, M^4/R_m, etc., do not contribute to the force on the sphere as they produce equal and opposite forces on the front and rear faces. Again it may be concluded that the main effect of the perturbation term ψ_0^1 is to retard the fluid motion close to the sphere, consequently reducing the viscous stresses and hence the drag. If the Reynolds number R is of the same order as M^2 then (52) should be modified as follows:

$$F_D = -6\pi\rho_0\nu U_a(1 + \tfrac{3}{8}R - 8M^2/3150). \tag{53}$$

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