—NOTES—

NOTE ON AN ASYMPTOTIC PROPERTY OF SOLUTIONS TO A CLASS OF FREDHOLM INTEGRAL EQUATIONS*

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In the course of studying an elastostatic load-transfer problem[1] we have recently encountered a need for determining the asymptotic behavior (at large arguments) of the solution to an inhomogeneous integral equation of Fredholm's second kind on a semi-infinite interval from known asymptotic properties of its kernel and right-hand member. Since the results obtained are apt to be of interest in connection with other applications and are, so far as we are aware, not available in the literature, a separate note on this issue may serve a useful purpose.

Consider the integral equation

$$\varphi(x) - \int_0^\infty K(x, s)\varphi(s)\, ds = f(x) \qquad (0 \leq x < \infty) \tag{1}$$

and let the kernel be of the translation-type, so that

$$K(x, s) = \lambda_1 G(x - s) + \lambda_2 G(x + s) \qquad (0 \leq x < \infty, 0 \leq s < \infty, x \neq s), \quad (2)^2$$

in which λ_1 and λ_2 are real constants. Further, let the real-valued given functions G and f have the following properties:

$$\left.\begin{array}{l} \text{(a) } G \text{ is continuous on } (0, \infty) \text{ and } G(-x) = G(x) \text{ for every } x \neq 0; \\ \text{(b) } G \text{ is absolutely integrable on } [0, \infty); \\ \text{(c) } G(x) = a/x^m + O(x^{-m-1}) \text{ as } x \to \infty \;\; (a \neq 0,\, 1 < m < \infty); \end{array}\right\} \tag{3}$$

$$\left.\begin{array}{l} \text{(a) } f \text{ is continuous on } [0, \infty); \\ \text{(b) } f(x) = b/x^n + o(x^{-n}) \text{ as } x \to \infty \;\; (b \neq 0,\, 0 < n < \infty). \end{array}\right\} \tag{4}$$

Note that G may become unbounded as $x \to 0$, provided its singularity at the origin is

* Received October 8, 1969. The results communicated here were obtained as part of an investigation conducted under Contract Nonr-220(58) with the Office of Naval Research in Washington, D. C.

[1] E. Sternberg and R. Muki, "Load-absorption by a filament in a fiber-reinforced material," Technical Report No. 20, Contract Nonr-220(58), California Institute of Technology; to appear in Z .angew. Math. Physik.

[2] There is no difficulty in generalizing the subsequent analysis to

$$K(x, s) = \lambda_1 G_1(x - s) + \lambda_2 G_2(x + s),$$

where G_1 and G_2 are distinct functions.

absolutely integrable. The real constants a, b, m, n, which govern the asymptotic behavior of G and f at infinity[3], are hereafter to be regarded as known.

Our ultimate objective is the

THEOREM. *Let G and f satisfy hypotheses* (3) *and* (4). *Let φ be a solution of* (1), (2), *such that*

$$\left.\begin{array}{l}\text{(a) } \varphi \text{ is continuous on } [0, \infty),\\ \text{(b) } \varphi(x) = \alpha/x^{\mu} + o(x^{-\mu}) \text{ as } x \to \infty \quad (\alpha \neq 0, 0 \leq \mu < \infty),\end{array}\right\} \tag{5}$$

where α and μ are real constants. Then

$$\alpha = b/\beta, \qquad \mu = n \quad \text{if} \quad m > n, \tag{6}$$

provided

$$\beta = 1 - 2\lambda_1 \int_0^\infty G(s)\, ds \neq 0. \tag{7}$$

Further,

$$\varphi(x) = O(x^{-m}) \quad \text{as } x \to \infty \text{ if } m \leq n. \tag{8}$$

Our proof of this theorem, given later on, depends crucially on a knowledge of the asymptotic character of the convolutions of G and φ entering (1) under various assumptions regarding the behavior, as $x \to \infty$, of G and φ themselves. This leads us to turn first to the subsequent

LEMMA. *Let G and φ be functions obeying* (3) *and* (5), *respectively. Let*

$$I_1(x) = \int_0^\infty \varphi(s)G(x - s)\, ds, \qquad I_2(x) = \int_0^\infty \varphi(s)G(x + s)\, ds \qquad (0 \leq x < \infty). \tag{9}$$

Then, as $x \to \infty$,

$$\left.\begin{array}{l} I_1(x) = \dfrac{2\alpha}{x^{\mu}} \displaystyle\int_0^\infty G(s)\, ds + o(x^{-\mu}) \quad \text{if} \quad \mu < m,\\[2ex] I_1(x) = \dfrac{1}{x^{m}} \displaystyle\int_0^\infty [2\alpha G(s) + a\varphi(s)]\, ds + o(\bar{x}^{m}) \quad \text{if} \quad \mu = m,\\[2ex] I_1(x) = \dfrac{a}{x^{m}} \displaystyle\int_0^\infty \varphi(s)\, ds + o(x^{-m}) \quad \text{if} \quad \mu > m, \end{array}\right\} \tag{10}$$

$$I_2(x) = o(x^{-\mu}) \quad \text{if} \quad \mu < m, \qquad I_2(x) = \frac{a}{x^{m}} \int_0^\infty \varphi(s)\, ds + o(x^{-m}) \quad \text{if} \quad \mu \geq m. \tag{11}$$

Proof. With a view toward establishing (10), choose numbers ϵ and x_0 subject to

$$1/m < \epsilon < 1, \qquad 1 < \log x_0 < x_0^{\epsilon} < x_0/2 \tag{12}$$

and, for every $x > x_0$, decompose the range of integration in the first of (9) into the six subintervals $[0, \log x]$, $[\log x, x^{\epsilon}]$, $[x^{\epsilon}, x - x^{\epsilon}]$, $[x - x^{\epsilon}, x]$, $[x, x + x^{\epsilon}]$, and $[x + x^{\epsilon}, \infty)$. This decomposition, upon recourse to (3), (5) and after elementary changes in the variable

[3] Condition (3c) may be relaxed by replacing the term $O(x^{-m-1})$ with $O(x^{-m-\delta})$, $\delta > 0$.

of integration, justifies the identity

$$I_1(x) = G(x) \int_0^{\log x} \varphi(s)\, ds + 2\varphi(x) \int_0^{x^\epsilon} G(s)\, ds + \sum_{i=1}^{6} \Omega_i(x) \qquad (x_0 < x < \infty), \tag{13}$$

where

$$\left.\begin{aligned} \Omega_1(x) &= \int_0^{\log x} \varphi(s)[G(x-s) - G(x)]\, ds, \qquad \Omega_2(x) = \int_{\log x}^{x^\epsilon} \varphi(s) G(x-s)\, ds, \\ \Omega_3(x) &= \int_{x^\epsilon}^{x - x^\epsilon} \varphi(s) G(x-s)\, ds, \qquad \Omega_4(x) = \int_0^{x^\epsilon} [\varphi(x-s) - \varphi(x)] G(s)\, ds, \\ \Omega_5(x) &= \int_0^{x^\epsilon} [\varphi(x+s) - \varphi(x)] G(s)\, ds, \qquad \Omega_6(x) = \int_{x^\epsilon}^{\infty} \varphi(x+s) G(s)\, ds. \end{aligned}\right\} \tag{14}$$

In view of (5), φ is bounded on $[0, \infty)$, i.e., there is an $M > 0$ such that

$$|\varphi(x)| < M \qquad (0 \le x < \infty). \tag{15}$$

Consequently (3b) and (3c), together with (5), at once yield the estimates[4]

$$\left.\begin{aligned} G(x) \int_0^{\log x} \varphi(s)\, ds &= o(x^{-1}) \qquad (0 \le \mu < \infty), \\ G(x) \int_0^{\log x} \varphi(s)\, ds &= \frac{a}{x^m} \int_0^{\infty} \varphi(s)\, ds + o(x^{-m}) \qquad (1 < \mu < \infty), \\ 2\varphi(x) \int_0^{x^\epsilon} G(s)\, ds &= \frac{2\alpha}{x^\mu} \int_0^{\infty} G(s)\, ds + o(x^{-\mu}) \qquad (0 \le \mu < \infty). \end{aligned}\right\} \tag{16}$$

Eqs. (13) and (16) clearly imply (10), provided

$$\Omega_i(x) = o(x^{-m}) + o(x^{-\mu}) \qquad (i = 1, 2, \cdots 6), \qquad (1 < m < \infty, 0 \le \mu < \infty), \tag{17}$$

which we now proceed to demonstrate.

To this end we infer from (3), (12), (14), (15) that

$$|\Omega_1(x)| < M \int_0^{\log x} |G(x-s) - G(x)|\, ds \le M \log x \max_{0 \le s \le \log x} |G(x-s) - G(x)| \qquad (x_0 < x < \infty). \tag{18}$$

On the other hand, because of (3c),

$$|G(x-s) - G(x)| = a/(x-s)^m - a/x^m + O(x^{-m-1}) \qquad (0 \le s \le \log x,\ x_0 < x < \infty) \tag{19}$$

and (18), (19) imply

$$\Omega_1(x) = o(x^{-m}) \qquad (0 \le \mu < \infty). \tag{20}$$

Next, according to (3c) and (5b), there is an $x_* > x_0$ and a positive constant A, such that

$$|\varphi(s) G(x-s)| < A/(s^\mu (x-s)^m) \qquad (\log x \le s \le x - \log x,\ x_* < x < \infty). \tag{21}$$

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[4] All order-of-magnitude statements are henceforth understood to refer to the limit as $x \to \infty$.

Thus, in view of the second of (14), one has

$$|\Omega_2(x)| < \frac{A}{(x - x^\epsilon)^m} \int_{\log x}^{x^\epsilon} (ds/s^\mu) \qquad (x_* < x < \infty). \tag{22}$$

Carrying out the integration in (22) and bearing in mind that $0 < \epsilon < 1$, $m > 1$, one confirms readily that

$$\Omega_2(x) = o(x^{-\mu}) \qquad (0 \leq \mu \leq 1), \qquad \Omega_2(x) = o(x^{-m}) \qquad (1 < \mu < \infty). \tag{23}$$

Proceeding to an estimate of Ω_3, we note from (14) and (21) that

$$|\Omega_3(x)| < A \int_{x^\epsilon}^{x - x^\epsilon} \frac{ds}{s^\mu (x - s)^m} < \frac{A}{x^{\epsilon m}(x - x^\epsilon)^m} \int_{x^\epsilon}^{x - x^\epsilon} s^{m-\mu}\, ds \qquad (x_* < x < \infty). \tag{24}$$

Evaluating the integral in (24) and observing from (12) that $\epsilon m > 1$, one arrives at

$$\Omega_3(x) = o(x^{-\mu}) \qquad (0 \leq \mu < m + 1),$$
$$\Omega_3(x) = o(x^{-m}) \qquad (m + 1 \leq \mu < \infty). \tag{25}$$

As far as Ω_4 and Ω_5 are concerned, (14) and (3), (5) yield the bounds

$$\left.\begin{aligned} |\Omega_4(x)| &\leq \max_{0 \leq s \leq x^\epsilon} |\varphi(x - s) - \varphi(x)| \int_0^\infty |G(s)|\, ds \qquad (x_0 < x < \infty), \\ |\Omega_5(x)| &\leq \max_{0 \leq s \leq x^\epsilon} |\varphi(x + s) - \varphi(x)| \int_0^\infty |G(s)|\, ds \qquad (x_0 < x < \infty), \end{aligned}\right\} \tag{26}$$

from which, by virtue of (5b), follows

$$\Omega_4(x) = o(x^{-\mu}), \qquad \Omega_5(x) = o(x^{-\mu}) \qquad (0 \leq \mu < \infty). \tag{27}$$

Finally,

$$\Omega_6(x) = o(x^{-\mu}) \qquad (0 \leq \mu < \infty), \tag{28}$$

as may be inferred directly from the last of (14) in conjunction with (3) and (5). Combining (20), (23), (25), (27), and (28), one sees that (17) holds true and thus the proof of (10) is complete.

It remains to verify (11). For this purpose, we first draw from (9), (12) the identity

$$\begin{aligned} I_2(x) = G(x) \int_0^{\log x} \varphi(s)\, ds + \int_0^{\log x} \varphi(s)[G(x + s) - G(x)]\, ds \\ + \int_{\log x}^\infty \varphi(s) G(x + s)\, ds \qquad (x_0 < x < \infty). \end{aligned} \tag{29}$$

The asymptotic behavior of the leading term in the right-hand member of (29) is known already from (16). Further, a procedure similar to that used in estimating[5] Ω_1 yields

$$\int_0^{\log x} \varphi(s)[G(x + s) - G(x)]\, ds = o(x^{-m}) \qquad (0 \leq \mu < \infty). \tag{30}$$

[5] See (18), (19).

Next, (3) and (5) assure the existence of positive numbers x_* and A, such that

$$\left|\int_{\log x}^{\infty} \varphi(s)G(x+s)\,ds\right| < A\int_{\log x}^{\infty} \frac{ds}{s^{\mu}(x+s)^{m}} \qquad (x_* < x < \infty). \tag{31}$$

An estimate of the right-hand side of (31) for $m > 1$ gives

$$\int_{\log x}^{\infty} \varphi(s)G(x+s)\,ds = \begin{cases} o(x^{-\mu}) & (0 \le \mu \le 1), \\ o(x^{-m}) & (1 < \mu < \infty). \end{cases} \tag{32}$$

Now (29), together with (16), (30), and (32), implies (11). This completes the proof of the lemma in its entirety.

We are now in a position to turn to a

Proof of the theorem. Consider first the case in which $m > n$. Suppose in this instance $0 \le \mu < n$. Then (1) and (4), (5) imply

$$\lim_{x\to\infty}\left[x^{\mu}\int_0^{\infty} K(x, s)\varphi(s)\,ds\right] = \alpha, \tag{33}$$

while from (2) and the lemma,

$$\lim_{x\to\infty}\left[x^{\mu}\int_0^{\infty} K(x, s)\varphi(s)\,ds\right] = 2\lambda_1\alpha\int_0^{\infty} G(s)\,ds. \tag{34}$$

But (33) contradicts (34) since, by hypothesis, (7) holds and $\alpha \neq 0$. Hence $\mu \ge n$. Suppose next that $\mu > n$. Then (1) and (4), (5) lead to

$$\lim_{x\to\infty}\left[x^{n}\int_0^{\infty} K(x, s)\varphi(s)\,ds\right] = -b, \tag{35}$$

whereas (2) and the lemma furnish

$$\lim_{x\to\infty}\left[x^{n}\int_0^{\infty} K(x, s)\varphi(s)\,ds\right] = 0, \tag{36}$$

which is incompatible with (35) because $b \neq 0$. It follows that

$$\mu = n \quad \text{if } m > n. \tag{37}$$

To confirm the first of (6) as well, multiply both sides of (1) by x^n and proceed to the limit as $x \to \infty$, invoking (37), (3), (4), (5), and the lemma. This yields

$$\alpha\left[1 - 2\lambda_1\int_0^{\infty} G(s)\,ds\right] = b, \tag{38}$$

and (38), in view of (7), assures that

$$\alpha = b/\beta \quad \text{if } m > n. \tag{39}$$

Finally, consider the case in which $m \le n$. Suppose $\mu < m$. Arguing as before, one finds that the mutually contradictory equations (33), (34) hold true once again. Hence $\mu \ge m$ and this conclusion, in conjunction with hypothesis (5b), validates (8). The proof of the theorem is now complete.

Acknowledgment. The authors are greatly obliged to J. K. Knowles for valuable discussions.