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LOWER BOUNDS ON WORK IN LINEAR VISCOELASTICITY*

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1. Introduction. Consider isothermal deformations of a linear viscoelastic material in simple tension or compression. Let $\sigma(t)$ and $\epsilon(t)$ denote, respectively, the stress and infinitesimal strain components at time t. The mechanical behavior of the material is taken to be governed by the constitutive equation [1]

$$\sigma(t) = \int_0^t G(t - \tau)\dot{\epsilon}(\tau)\, d\tau, \tag{1}$$

where $G(t)$ is the relaxation modulus defined for nonnegative times. The material is taken to be in the unstressed and unstrained virgin state for $t \leq 0$.

The work done on the material during the time interval $(0, T)$ is given by

$$W = \int_0^T \sigma(t)\dot{\epsilon}(t)\, dt, \tag{2}$$

which can be written as the second order functional

$$W = \int_0^T \int_0^t G(t - \tau)\dot{\epsilon}(\tau)\dot{\epsilon}(t)\, d\tau\, dt. \tag{3}$$

Sufficient conditions on $G(t)$, which guarantee that $W > 0$, have been given by Breuer and Onat in [2]. Recently, Martin and Ponter [3] have inquired into the possibility of determining a *positive* lower bound for W. They found that if

$$G(t) = \sum_{i=1}^n c_i e^{-a_i t}, \qquad c_i > 0, \qquad a_i > 0, \tag{4}$$

then W is bounded from below by

$$W \geq (1/T)\bar{G}(2/T)\epsilon^2(T), \tag{5}$$

where $\bar{G}(s)$ is the Laplace transform of $G(t)$. In view of (4), we may write (5) in the form

$$W \geq \sum_{i=1}^n \frac{c_i}{2 + a_i T}\, \epsilon^2(T), \tag{6}$$

which is more suitable for our purposes. As pointed out in [3], (6) provides a positive

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lower bound on the work required to produce a given strain in a viscoelastic material at time T in terms of T and $\epsilon(T)$ and independent of the history of strain.

We shall rederive the result (6) by a very elementary method, using no stronger tool than the Schwarz inequality. Our method of proof may then be generalized so as to obtain an inequality of the type (6) for a much wider class of relaxation moduli. It does not appear that such generalization can be achieved along the lines of the method used in [3].

2. The result of Martin and Ponter. In order to derive (6), it is clearly sufficient to consider the relaxation modulus

$$G(t) = e^{-at}, \qquad a > 0, \tag{7}$$

and show that

$$W = \int_0^T \int_0^t e^{-a(t-\tau)}\dot{\epsilon}(\tau)\dot{\epsilon}(t)\, d\tau\, dt \geq \frac{1}{2 + aT}\, \epsilon^2(T). \tag{8}$$

Since $c_i > 0$, (6) will follow.

Defining

$$\sigma(t) = \int_0^t e^{-a(t-\tau)}\dot{\epsilon}(\tau)\, d\tau, \tag{9}$$

and differentiating (9) with respect to t, we obtain

$$\dot{\sigma}(t) = \dot{\epsilon}(t) - a\sigma(t). \tag{10}$$

Integrating (10) and using the fact that $\sigma(0) = \epsilon(0) = 0$, we reach

$$\epsilon(t) = \sigma(t) + a \int_0^t \sigma(\tau)\, d\tau. \tag{11}$$

From (8), (9) and (10) follows now

$$W = \frac{1}{2}\, \sigma^2(T) + a \int_0^T \sigma^2(t)\, dt. \tag{12}$$

Next we define

$$X_1 = \sigma(T), \qquad X_2 = \int_0^T \sigma(t)\, dt, \tag{13}$$

and, using the Schwarz inequality, we find

$$X_2^2 \leq T \int_0^T \sigma^2(t)\, dt. \tag{14}$$

Defining

$$H = (1/2)X_1^2 + (a/T)X_2^2\,, \tag{15}$$

we find from (12)–(15) that

$$W \geq H. \tag{16}$$

Next we find, on the basis of (11) and (13), that

$$\epsilon^2(T) = (X_1 + aX_2)^2. \tag{17}$$

We consider, therefore, the quadratic form

$$F = H - \lambda\epsilon^2(T), \tag{18}$$

where λ is a real parameter.

The well known condition that $F \geq 0$ for all X_1 and X_2 is

$$\Delta = \begin{vmatrix} 1/2 - \lambda & -\lambda a \\ -\lambda a & a/T - \lambda a^2 \end{vmatrix} = \frac{a}{2T}\,[1 - \lambda(2 + aT)] \geq 0. \tag{19}$$

That is,

$$\lambda \leq 1/(2 + aT), \tag{20}$$

will guarantee that $F \geq 0$. Indeed, when equality holds in (20) we find

$$F = \frac{aT}{2}\left(\frac{1}{2 + aT}\right)\left(X_1 - \frac{2}{T}X_2\right)^2 \geq 0. \tag{21}$$

Thus and by (16) and (18) we see that (8) is fulfilled.

3. Generalization of the result. We consider next a generalization of the method employed in Sec. 2. Consider

$$\sigma(t) = \int_0^t G(t - \tau)\dot{\epsilon}(\tau)\,d\tau, \tag{22}$$

where $G(t)$ is three times continuously differentiable on every finite interval $[0, T]$, and where, for simplicity, $G(0) = 1$. We are also assuming throughout that $\sigma(0) = \epsilon(0) = 0$. Equation (22) has the inverse

$$\epsilon(t) = \int_0^t J(t - \tau)\dot{\sigma}(\tau)\,d\tau, \tag{23}$$

where $J(t)$ is the creep compliance, related to $G(t)$ by the well-known equation

$$\int_0^t G(t - \tau)J(\tau)\,d\tau = t. \tag{24}$$

Differentiating (24) repeatedly and evaluating the results at $t = 0$, we obtain

$$J(0) = G(0) = 1, \qquad J'(0) = -G'(0). \tag{25}$$

Moreover, the assumptions on $G(t)$ guarantee the existence of $J''(t)$ on every finite interval $[0, T]$.

We now make the following definition.

Definition. The function $G(t)$ will be called an *admissible relaxation modulus* if

(1) $G(t)$ is three times continuously differentiable on every finite interval $[0, T]$.
(2) $G(0) = 1$, $G'(0) < 0$.
(3) $J''(t)$ is of positive type, where $J(t)$ is defined by (24).

We recall that $P(t)$ is of positive type if for every continuous function $y(t)$ which is not identically zero, we have

$$\int_0^T \int_0^t P(t - \tau)y(\tau)y(t)\,d\tau\,dt \geq 0, \qquad T > 0. \tag{26}$$

We are now in a position to state the following theorem.

Theorem. *Let $G(t)$ be an admissible relaxation modulus, and let $\sigma(0) = \epsilon(0) = 0$. If W is defined as in* (3), *we have*

$$W \geq \frac{1}{2 + \beta^2 T} \epsilon^2(T), \tag{27}$$

where

$$\beta^2 = (1/J'(0))[J'(0) + J''(0)T]^2 > 0. \tag{28}$$

Before proving the theorem, we note that if $G(t) = e^{-at}$ with $a > 0$, then $J(t) = 1 + at$, $\beta^2 = a$, and (27) reduces to (8).

Proof of the Theorem: Differentiating (23) and using (25), we obtain

$$\dot{\epsilon}(t) = \dot{\sigma}(t) + \int_0^t J'(t - \tau)\dot{\sigma}(\tau)\, d\tau. \tag{29}$$

Integrating the last term by parts, we reach

$$\dot{\epsilon}(t) = \dot{\sigma}(t) + J'(0)\sigma(t) + \int_0^t J''(t - \tau)\sigma(\tau)\, d\tau. \tag{30}$$

Integrating (30) we have

$$\epsilon(T) = \sigma(T) + J'(0) \int_0^T \sigma(t)\, dt + \int_0^T \int_0^t J''(t - \tau)\sigma(\tau)\, d\tau\, dt. \tag{31}$$

Therefore, and since $J'(0) > 0$ by (25) and the definition of an admissible relaxation modulus,

$$|\epsilon(T)| \leq |\sigma(T)| + J'(0) \int_0^T |\sigma(t)|\, dt + \left| \int_0^T \int_0^t J''(t - \tau)\sigma(\tau)\, d\tau\, dt \right| \cdot \tag{32}$$

Since $J''(t)$ is of positive type, we know [4] that $J''(0) \geq |J''(t)|$ for all $t \geq 0$. Consequently,

$$\begin{aligned} \left| \int_0^T \int_0^t J''(t - \tau)\sigma(\tau)\, d\tau\, dt \right| &\leq J''(0) \int_0^T \int_0^t |\sigma(\tau)|\, d\tau\, dt \\ &\leq J''(0) \int_0^T \int_0^T |\sigma(\tau)|\, d\tau\, dt = J''(0)T \int_0^T |\sigma(t)|\, dt. \end{aligned} \tag{33}$$

Defining

$$Y_1 = |\sigma(T)|, \qquad Y_2 = \int_0^T |\sigma(t)|\, dt, \tag{34}$$

we find from (32)–(34) that

$$\epsilon^2(T) \leq [Y_1 + (J'(0) + J''(0)T)Y_2]^2. \tag{35}$$

On the other hand, (30) and (2) yield

$$W = \frac{1}{2} \sigma^2(T) + J'(0) \int_0^T \sigma^2(t)\, dt + \int_0^T \int_0^t J''(t - \tau)\sigma(\tau)\sigma(t)\, d\tau\, dt, \tag{36}$$

and, using again the Schwarz inequality and the fact that $J''(t)$ is of positive type, we reach

$$W \geq M, \tag{37}$$

where

$$M = (1/2)Y_1^2 + (J'(0)/T)Y_2^2, \tag{38}$$

Y_1 and Y_2 being defined by (34).

Using the same procedure as before, we find that

$$\mu \leq \frac{1}{2 + \beta^2 T}, \tag{39}$$

will guarantee that

$$M - \mu\epsilon^2(T) \geq 0. \tag{40}$$

Indeed, when equality holds in (39), we find

$$M - \mu\epsilon^2(T) \geq \frac{\beta^2 T}{2}\left(\frac{1}{2 + \beta^2 T}\right)\left(Y_1 - \frac{2}{T}\left(\frac{J'(0)}{\beta^2}\right)^{1/2} Y_2\right)^2 \geq 0. \tag{41}$$

Equation (27) now follows from (37), (39) and (41), completing the proof.

It is clear that the restriction $G(0) = 1$ in the definition of an admissible relaxation modulus leads to no loss in generality. If $G(0) \neq 1$, *as long as it is positive,* we merely multiply the right-hand side of (27) by $G(0)$. Moreover, we can obtain similar results for general deformations of linear isotropic and anisotropic solids provided the relaxation moduli are admissible in the sense of our definition.

4. Examples. In this section we give three examples of the foregoing considerations. Let the constants a_1, a_2, c_1, c_2 satisfy the following inequalities:

$$c_1 > c_2 > 0, \qquad c_1 a_1 > c_2 a_2 > 0, \qquad c_1 a_2 > c_2 a_1 > 0, \qquad a_1 \neq a_2. \tag{42}$$

In particular, all four constants are positive, and since $c_1 > c_2$, (42) does not restrict the relative magnitudes of a_1 and a_2.

Next, define $G(t)$ by

$$G(t) = \frac{1}{c_1 - c_2}(c_1 e^{-a_1 t} - c_2 e^{-a_2 t}). \tag{43}$$

We shall take (43) as the basis for our illustrations. Before we do this, however, we observe that if we extend the definition of $G(t)$ by the requirement $G(-t) = G(t)$ for all t, we obtain for the Fourier transform of $G(t)$ the expression

$$\hat{G}(u) = \frac{1}{c_1 - c_2}\left(\frac{2}{\pi}\right)^{1/2}\left[\frac{(c_1 a_1 - c_2 a_2)u^2 + a_1 a_2(c_1 a_2 - c_2 a_1)}{(a_1^2 + u^2)(a_2^2 + u^2)}\right] > 0, \tag{44}$$

so that according to the results obtained in [2] the corresponding W defined by (3) is certainly positive. We are, however, in the position to find a positive lower bound for W. It is readily verified that corresponding to (43) we obtain from (24)

$$J(t) = A_1 + A_2 t + A_3 e^{-\alpha t}, \tag{45}$$

where

$$\alpha = \frac{c_1 a_2 - c_2 a_1}{c_1 - c_2} > 0, \qquad A_1 = \frac{\alpha(a_1 + a_2) - a_1 a_2}{\alpha^2},$$
$$A_2 = \frac{a_1 a_2}{\alpha}, \qquad A_3 = \frac{c_1 c_2}{\alpha^2} \frac{(a_1 - a_2)^2}{(c_1 - c_2)^2} > 0 \tag{46}$$

One can readily verify that $G(t)$ as defined by (43) subject to (42) is *admissible.* Indeed, $G(0) = 1$, $G'(0) = (c_2 a_2 - c_1 a_1)/(c_1 - c_2) < 0$ by (42), while $J''(t) = \alpha^2 A_3 e^{-\alpha t}$ is of positive type by the results of Sec. 2 and since $A_3 > 0$, $\alpha > 0$. Moreover, the results of Sec. 2 *do not cover* this case, because of the *minus sign* in front of the second exponential in (43)

We now distinguish three cases.

Case I. $a_1 < a_2$ and $(a_1/a_2)^2(c_1/c_2) > 1$.

It is easily seen in this case that (42) imply

$$G(t) > 0, \qquad G'(t) < 0, \qquad G''(t) > 0, \quad \text{for all} \quad t \geq 0. \tag{47}$$

This is a rather conventional relaxation modulus, decreasing steadily to zero as t increases and being convex from *below* for all $t \geq 0$. For example, we may take

$$G(t) = \tfrac{1}{3}(4e^{-2t} - e^{-3t}). \tag{48}$$

Correspondingly, we find

$$J(t) = (1/25)(24 + 45t + e^{-(10/3)t}), \tag{49}$$

with

$$J'(0) = 5/3, \qquad J''(0) = 4/9, \tag{50}$$

and finally,

$$W \geq \frac{1}{2 + \frac{3}{5}(\frac{5}{3} + \frac{4}{9}T)^2 T} \epsilon^2(T). \tag{51}$$

Case II. $a_1 < a_2$ and $(a_1/a_2)^2(c_1/c_2) \leq 1$.

Again we find that

$$G(t) > 0, \qquad G'(t) < 0, \quad \text{for all} \quad t \geq 0, \tag{52}$$

but

$$G''(t) < 0 \qquad \text{for} \quad 0 \leq t < \frac{1}{a_2 - a_1} \log\left[\left(\frac{a_2}{a_1}\right)^2 \left(\frac{c_2}{c_1}\right)\right],$$
$$G''(t) > 0 \qquad \text{for} \qquad t > \frac{1}{a_2 - a_1} \log\left[\left(\frac{a_2}{a_1}\right)^2 \left(\frac{c_2}{c_1}\right)\right]. \tag{53}$$

In illustration, consider

$$G(t) = \tfrac{1}{3}(4e^{-2t} - e^{-5t}), \tag{54}$$

so that

$$J(t) = (1/9)(8 + 15t + e^{-6t}), \tag{55}$$

$$J'(0) = 1, \qquad J''(0) = 4. \tag{56}$$

Finally,

$$W \geq \frac{1}{2 + (1 + 4T)^2 T} \epsilon^2(T). \tag{57}$$

This is a somewhat unexpected result. The curve (54) decreases steadily to zero as t increases, but it is *convex from above* up to the point $t = (2/3) \log (5/4)$. This point is a point of inflection, beyond which the curve remains convex from *below*.

Case III. $a_1 > a_2$.

This case is even more surprising than the previous one. Upon using (42), we find that the curve decreases steadily to a *negative* minimum value, being convex from *below*. It never rises above the t-axis after it has crossed it once on the way down to its minimum. Instead, it rises steadily from the minimum value through a point of inflection, beyond which it remains convex from *above* and approaches zero through negative values as t increases. For example, consider

$$G(t) = k + G_1(t), \tag{58}$$

where

$$G_1(t) = \tfrac{1}{3}(4e^{-2t} - e^{-t}), \qquad k > 1/48. \tag{59}$$

The addition of k keeps the curve above the t-axis. Its minimum value is $(k - 1/48)$ at $t = \log 8$. The point of inflection is at $t = \log 16$. Corresponding to $G_1(t)$ there is a W_1 which is, of course, itself bounded away from zero. Indeed, we find that $G_1(t)$ gives rise to $J_1(t)$, where

$$J_1(t) = 3t + e^{-(2/3)t}, \tag{60}$$

$$J_1'(0) = 7/3, \qquad J_1''(0) = 4/9, \tag{61}$$

so that

$$W_1 \geq \frac{1}{2 + \frac{3}{7}(\frac{7}{3} + \frac{4}{9}T)^2 T} \epsilon^2(T). \tag{62}$$

Finally,

$$W = \int_0^T \int_0^t k\dot{\epsilon}(\tau)\dot{\epsilon}(t)\, d\tau\, dt + W_1 \geq \left[\tfrac{1}{2}k + \frac{1}{2 + \frac{3}{7}(\frac{7}{3} + \frac{4}{9}T)^2 T}\right]\epsilon^2(T). \tag{63}$$

The foregoing three examples serve to illustrate the results of Sec. 3. Although the relaxation moduli chosen in the illustrations are somewhat unconventional owing to the minus sign in front of the second term, they are still very useful in two important respects. First, they serve to provide positive lower bounds on a wider class of functionals than those covered in Sec. 2, without interpreting them necessarily in terms of viscoelasticity theory. It is also clear that we can take for $G(t)$ a linear combination of any number of decreasing exponentials, with both positive and negative coefficients, and the results of Sec. 3 will hold so long as $G(t)$ is a sum of admissible relaxation moduli. Secondly, in determining a relaxation modulus experimentally, for a given material, it may well happen that a curve of the type (43) will better fit the data points than one of the type (4). Should the curve so fitted turn out to be admissible, the results obtained here may be applied.

Finally, it should be observed that, owing to the inequalities used to derive (27), the lower bound defined therein need not necessarily be the *greatest* lower bound.

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