—NOTES—

USEFUL STRAIN HISTORIES IN LINEAR VISCOELASTICITY*

By W. A. DAY** (*Carnegie-Mellon University*)

Suppose that a material has been subjected to a given strain history in the past with the result that energy has been stored in it. It is natural to ask if it is possible to extract part of this energy as useful work by taking the material through an appropriate path in strain space. If this is possible then we would like to know how much work can be extracted and which strain paths can be used to extract useful work. Questions of this sort have been considered previously for very special classes of one-dimensional linear viscoelastic materials by Breuer and Onat [1] and by Day [3]. This paper treats an extensive class of (three-dimensional) linear viscoelastic materials. Our purpose is to give simple conditions on a strain history guaranteeing that the history be *useful* in the sense that useful work can be extracted from it in closed connections of the history—processes connecting the history to its final value by a closed path. In addition we give a lower bound on the amount of useful work which can be recovered from a history meeting our condition and we construct closed connections extracting useful work.

To make these ideas precise we use the following notation and terminology. By the *strain space* ζ is meant the set of all symmetric linear transformations of R^3 into itself. The elements $\alpha, \beta, \gamma \cdots$ of ζ are to be interpreted as infinitesimal strain tensors and ζ is to be considered an inner product space with inner product $\alpha \cdot \beta = \text{trace } \alpha\beta$ and norm $||\alpha|| = (\alpha \cdot \alpha)^{1/2}$. We also consider the vector space $\mathfrak{L}(\zeta)$ of all linear transformations of ζ into itself endowed with the usual norm $||L|| = \sup \{||L\alpha|| : ||\alpha|| = 1\}$. We say that a measurable function f: $[0, \infty) \to \zeta$ is a *strain history* if it is continuous at 0 and we call $f(0)$ its *final value*. The constant function α^* with value α on $[0, \infty)$ is a simple example of a strain history. We call a function $\Phi : (-\infty, \infty) \to \zeta$ a *closed connection* of the strain history f if (1) $\Phi(t) = f(-t)$ for $-\infty < t \leq 0$, (2) there is a number $\tau > 0$ such that $\Phi(t) = f(0)$ for $t \geq \tau$ and (3) Φ is continuous and piecewise smooth on $[0, \infty)$. If Φ is a closed connection of f, then $\Phi(t) = \Phi(0) = f(0)$, for all sufficiently large numbers t. An example of a closed connection of f is provided by the *constant continuation* $\bar{f}$ with values $\bar{f}(t) = f(-t)$, for $-\infty < t < 0$, and $\bar{f}(t) = f(0)$, for $t \geq 0$. It should be noted that if Φ is a closed connection of f then, for each $t \geq 0$, the function $\Phi^t : [0, \infty) \to \zeta$ defined by $\Phi^t(u) = \Phi(t - u)$ is a strain history with final value $\Phi(t -) = \lim_{u \to t-} \Phi(u)$.

A linear viscoelastic material is determined by its *relaxation function* by which we mean any smooth function G: $[0, \infty) \to \mathfrak{L}(\zeta)$ with derivative $\dot{G}$: $[0, \infty) \to \mathfrak{L}(\zeta)$ and having the following properties: (1) $G(\infty) = \lim_{t \to \infty} G(t)$ exists and $\lim_{t \to \infty} \dot{G}(t) = 0$, (2) $\int_0^\infty t\dot{G}(t)\, dt < \infty$, (3) $G(0) - G(\infty)$ is symmetric and positive definite.[1] With each strain history f

* Received May 20, 1968.

** Present address: Hertford College, Oxford.

[1]Various authors have discussed 'thermodynamic' assumptions which ensure the symmetry and positive semidefiniteness of G(0) − G(∞). See, for example, Coleman [2], Day [3], Gurtin and Herrera [4] and Shu and Onat [5]. Our assumption that G(0) − G(∞) is actually positive definite rules out elastic materials since for those materials G is constant.

and each strain α in ζ (α is the 'present value') we associate a *stress* $s(f, \alpha)$ by means of the constitutive relation of linear viscoelasticity, namely

$$s(f, \alpha) = G(0)\alpha + \int_0^\infty \dot{G}(u)f(u)\,du,$$

provided the integral exists. The *equilibrium stress* for the strain α is

$$s^*(\alpha) = s(\alpha^*, \alpha) = G(\infty)\alpha.$$

Let f be a strain history and $\bar{f}$ its constant continuation defined previously. We say that f has the *stress relaxation property* if the function $t \to s(\bar{f}^t, f(0))$ is continuous on $[0, \infty)$ and

$$\lim_{t\to\infty} s(\bar{f}^t, f(0)) = s^*(f(0)).$$

Since

$$s(\bar{f}^t, f(0)) = G(t)f(0) + \int_0^\infty \dot{G}(u + t)f(u)\,du,$$

the history f has the stress relaxation property if and only if the function $t \to \int_0^\infty \dot{G}(u + t)f(u)\,du$ is continuous on $[0, \infty)$ and

$$\lim_{t\to\infty} \int_0^\infty \dot{G}(u + t)f(u)\,du = 0.$$

For closed connections Φ of f the *work* $w(\Phi)$ may be computed from the usual expression

$$\int_0^\infty s(\Phi^t, \Phi(t))\cdot\dot{\Phi}(t)\,dt,$$

whenever this integral exists, and if it happens that $w(\Phi) < 0$ we say, following Breuer and Onat [1], that *the material does useful work of amount* $-w(\Phi) > 0$. We say too that a strain history f is *useful* if the *maximum recoverable work*

$$W(f) = \sup\,\{-w(\Phi) : \Phi \text{ a closed connection of } f\}$$

is positive.

The following theorem is an assertion about strain histories f with $s(f, f(0)) \neq s^*(f(0))$. For an elastic material no history can meet this requirement and consequently we distinguish these histories by calling them *inelastic* with respect to the given viscoelastic material.

THEOREM. *If the strain history f is inelastic and has the stress relaxation property then it is useful, the maximum recoverable work $W(f)$ is bounded below by the positive number*

$$l(f) = \tfrac{1}{2}\,\|(G(0) - G(\infty))^{-1/2}(s(f, f(0)) - s^*(f(0)))\|^2$$

and there is a sequence Φ_n of closed connections of f with $-w(\Phi_n) \to l(f)$ as $n \to \infty$.

PROOF. If α in ζ is any strain we can define a sequence Φ_n of closed connections of

---

[2]Here $(G(0) - G(\infty))^{-1/2}$ is the unique positive definite and symmetric square root of $(G(0) - G(\infty))^{-1}$.

f, each of which is piecewise linear on $[0, \infty)$, by requiring that $\Phi_n^0 = f$ and

$$\begin{aligned} \Phi_n(u) &= f(0) + nu(\alpha - f(0)), & 0 \le u < 1/n, \\ &= \alpha, & 1/n \le u < n, \\ &= \alpha + \frac{1}{n}(u - n)(f(0) - \alpha), & n \le u < 2n, \\ &= f(0), & 2n \le u < \infty. \end{aligned}$$

The strain α is at our disposal and will be chosen in a convenient way later on.

Assuming that f has the stress relaxation property, the definition of work tells us that

$$\begin{aligned} w(\Phi_n) &= \left(\int_0^{1/n} + \int_0^{2n}\right) s(\Phi_n^t, \Phi_n(t)) \cdot \dot{\Phi}_n(t)\, dt \\ &= \left(n \int_0^{1/n} s(\Phi_n^t, \Phi_n(t))\, dt\right) \cdot (\alpha - f(0)) \\ &\quad + \left(\frac{1}{n} \int_n^{2n} s(\Phi_n^t, \Phi_n(t))\, dt\right) \cdot (f(0) - \alpha). \end{aligned}$$

A straightforward computation shows that

$$\begin{aligned} n \int_0^{1/n} s(\Phi_n^t, \Phi_n(t))\, dt &= \int_0^1 s(\Phi_n^{t/n}, \Phi_n(t/n))\, dt \\ &= \frac{1}{2} G(0)(\alpha + f(0)) + \int_0^\infty \dot{G}(u) f(u)\, du \\ &\quad + \int_0^1 \Bigg[\int_0^\infty (\dot{G}(u + t/n) - \dot{G}(u)) f(u)\, du \\ &\qquad + (G(t/n) - G(0))(f(0) + t(\alpha - f(0))) \\ &\qquad - \left(\frac{1}{n} \int_0^t u \dot{G}(u/n)\, du\right)(\alpha - f(0))\Bigg]\, dt \end{aligned}$$

and that

$$\begin{aligned} \frac{1}{n} \int_n^{2n} s(\Phi_n^t, \Phi_n(t))\, dt &= \int_0^1 s(\Phi_n^{nt+n}, \Phi_n(nt + n))\, dt \\ &= \tfrac{1}{2} G(\infty)(\alpha + f(0)) \\ &\quad + \int_0^1 \Bigg[\int_0^\infty \dot{G}(u + nt + n) f(u)\, du \\ &\qquad + (G(nt + n - 1/n) - G(\infty))\alpha \\ &\qquad + t(G(nt) - G(\infty))(f(0) - \alpha) \\ &\qquad + (G(nt + n) - G(nt + n - 1/n)) f(0) \\ &\qquad - \left(\frac{1}{n} \int_0^{nt} u \dot{G}(u)\, du\right)(f(0) - \alpha) \\ &\qquad + \left(n \int_0^{1/n} u \dot{G}(nt + n - u)\, du\right)(\alpha - f(0))\Bigg]\, dt. \end{aligned}$$

It follows from the properties of the function G and the assumption that f has the stress relaxation property that, as $n \to \infty$,

$$n \int_0^{1/n} s(\Phi_n^t, \Phi_n(t))\, dt \to \frac{1}{2} G(0)(\alpha + f(0)) + \int_0^\infty \dot{G}(u) f(u)\, du$$

and

$$\frac{1}{n} \int_n^{2n} s(\Phi_n^t, \Phi_n(t))\, dt \to \frac{1}{2} G(\infty)(\alpha + f(0)),$$

and so

$$\lim_{n\to\infty} w(\Phi_n) = -\frac{1}{2}(G(0) - G(\infty))(\alpha + f(0))\cdot(\alpha - f(0)) - (\alpha - f(0))\cdot\int_0^\infty \dot{G}(u) f(u)\, du.$$

If we introduce E, the symmetric and positive definite square root of $G(0) - G(\infty)$, this limit can be written as

$$\begin{aligned} \frac{1}{2} Ef(0)\cdot Ef(0) &+ Ef(0)\cdot E^{-1}\int_0^\infty \dot{G}(u)f(u)\, du - \frac{1}{2} E\alpha\cdot E\alpha - E\alpha\cdot E^{-1}\int_0^\infty \dot{G}(u)f(u)\, du \\ &= \frac{1}{2}\left\| Ef(0) + E^{-1}\int_0^\infty \dot{G}(u)f(u)\, du\right\|^2 - \frac{1}{2}\left\| E\alpha + E^{-1}\int_0^\infty \dot{G}(u)f(u)\, du\right\|^2 \\ &= \tfrac{1}{2}\, \| E^{-1}(s(f, f(0)) - s^*(f(0)))\|^2 - \tfrac{1}{2}\, \| E^{-1}(s(f, \alpha) - s^*(\alpha))\|^2. \end{aligned}$$

By choosing

$$\alpha = -(G(0) - G(\infty))^{-1}\int_0^\infty \dot{G}(u)f(u)\, du$$

we ensure that $s(f, \alpha) = s^*(\alpha)$ and so, with this choice,

$$-\lim_{n\to\infty} w(\Phi_n) = \tfrac{1}{2}\, \| E^{-1}(s(f, f(0)) - s^*(f(0))\|^2,$$

which proves the theorem.

Remark. For the one-dimensional Maxwell material with relaxation function $G(t) = ae^{-bt}$ ($a, b > 0$) the maximum recoverable work can be computed explicitly (see [1] and [3]) and has the value

$$\begin{aligned} W(f) &= (1/2)a\left[f(0) - b\int_0^\infty e^{-bt} f(t)\, dt\right]^2 \\ &= (1/2)[(G(0) - G(\infty))^{-1/2}(s(f, f(0)) - s^*(f(0)))]^2. \end{aligned}$$

In other words, the lower bound given in the theorem is actually equal to the maximum recoverable work in this special case and the sequence Φ_n ultimately extracts all the recoverable work.

Acknowledgements. I thank Professor M. E. Gurtin for his comments on an earlier draft of this paper.

References

[1] S. Breuer and E. T. Onat, *On recoverable work in linear viscoelasticity*, Z. Angew. Math. Phys. **15**, 12–21 (1964)

[2] B. D. Coleman, *On thermodynamics, strain impulses and viscoelasticity,* Arch. Rational Mech. Anal. **17**, 230–254 (1964)

[3] W. A. Day, *Thermodynamics based on a work axiom,* Arch. Rational Mech. Anal. **31**, 1–34 (1968)

[4] M. E. Gurtin and I. Herrera, *On dissipation inequalities and linear viscoelasticity,* Quart. Appl. Math. **23**, 235–245 (1965)

[5] L. S. Shu and E. T. Onat, *On anisotropic linear viscoelastic solids,* Proceedings of the Fourth Symposium on Naval Structural Mechanics, Purdue University, April 1965, reprinted in *Mechanics and chemistry of solid propellants*, Pergamon, New York, 1966