A NOTE ON USEFUL WORK*

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To begin with, we discuss *isothermal* processes in a body and denote the deformation gradient tensor and the Piola–Kirchhoff stress tensor at a material point at time t by $e(t)$ and $s(t)$, respectively. For simplicity we take all functions of time to be smooth and denote time differentiation by a superposed $\cdot$. A particular material is described by specifying the functional relationship

$$s(t) = S(e^t), \tag{1}$$

between the stress at time t and e^t, the history of the deformation gradient up to t, which is defined on $[0, \infty)$ by

$$e^t(u) = e(t - u), \qquad 0 \leq u < \infty. \tag{2}$$

Once the functional S is specified the work done in the process $e(\cdot)$ between times 0 and τ can be found from the usual expression

$$w(e(\cdot), \tau) = \int_0^\tau \text{trace}\,(s(t)\dot{e}(t)^T)\, dt, \tag{3}$$

where $\dot{e}(t)^T$ is the transpose of $\dot{e}(t)$.

Suppose that the material is subjected to a given deformation gradient history f, say, up to time $t = 0$. Let us consider processes $e(\cdot)$ which are *closed connections of* f in the sense that $e^0 = f$ and $e(\cdot)$ assumes the value $e(0) = f(0)$ at some later time τ; i.e. $e(\tau) = f(0)$ for some $\tau > 0$, and for each closed connection of f let us compute the work $w(e(\cdot), \tau)$. If it happens that $w(e(\cdot), \tau) < 0$ we shall say, following Breuer and Onat [1], that *the material does useful work of amount* $-w(e(\cdot), \tau) > 0$. These definitions lead us to ask the following question: if f is given and if $W > 0$ is any quantity of work, no matter how large, can we choose a closed connection $e(\cdot)$ of f so that the material does an amount of useful work exceeding W, i.e., $-w(e(\cdot), \tau) > W$?

The purpose of this note is to point out that whenever two thermodynamic properties T1 and T2 hold the answer to the question posed above is 'No'. The statement of T1 and T2 requires that we introduce the free energy $\psi(\cdot)$ determined by $e(\cdot)$ through the constitutive relation

$$\psi(t) = \Psi(e^t). \tag{4}$$

The thermodynamic properties are

T1. *In any process* $e(\cdot)$ *the work done between times* 0 *and* τ *is not less than the change in free energy; i.e.*

$$w(e(\cdot), \tau) \geq \psi(\tau) - \psi(0). \tag{5}$$

T2. *Among all processes* $e(\cdot)$ *with a given value* c *at* $t = \tau$ *the constant process with*

*Received May 20, 1968.

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value c has the least free energy; i.e. if $e(\tau) = c$, *then*

$$\Psi(e^\tau) \geq \Psi(c^*), \tag{6}$$

where c^* *is the constant history on* $[0, \infty)$ *with value c.*

Properties T1 and T2 were first established for a class of simple materials by Coleman [2] (see also Coleman and Mizel [3] and Gurtin [5]).

If T1 and T2 do hold and $e(\cdot)$ is any closed connection of a given history f with $e(\tau) = f(0)$, $\tau > 0$, then, by the definition of a closed connection,

$$\psi(0) = \Psi(e^0) = \Psi(f)$$

and, by T2,

$$\psi(\tau) = \Psi(e^\tau) \geq \Psi(f(0)^*).$$

An application of T1 now shows that

$$w(e(\cdot), \tau) \geq \Psi(f(0)^*) - \Psi(f),$$

or,

$$-w(e(\cdot), \tau) \leq \Psi(f) - \Psi(f(0)^*), \tag{7}$$

where, by T2 again,

$$\Psi(f) - \Psi(f(0)^*) \geq 0. \tag{8}$$

In other words we have shown that *if* T1 *and* T2 *hold and if* f *is a given history then in no closed connection of* f *can the material do an amount of useful work exceeding the difference* $\Psi(f) - \Psi(f(0)^*)$ *between the free energy* $\Psi(f)$ *of the history* f *and the free energy* $\Psi(f(0)^*)$ *of the constant history with value* $f(0)$.

Precisely the same conclusion holds for nonisothermal processes if by 'process' we understand the ordered deformation gradient-temperature pair $(e(\cdot), \theta(\cdot))$, if the constitutive relations for the stress, free energy and entropy are taken to be

$$s(t) = S(e^t, \theta^t), \qquad \psi(t) = \Psi(e^t, \theta^t), \qquad \eta(t) = H(e^t, \theta^t), \tag{9}$$

respectively, if the definition (3) of work is generalized to

$$w(e(\cdot), \theta(\cdot), \tau) = \int_0^\tau (\text{trace}\,(s(t)\dot{e}(t)^T) - \eta(t)\dot{\theta}(t))\, dt \tag{10}$$

and if T1 and T2 still hold. Properties T1 and T2 do hold in the theories cited previously ([2], [3], [5]) and so for the classes of materials considered in those theories the conclusion is valid.

In a forthcoming paper [4], I shall show that, for a broad class of materials, all the results connecting stress, entropy and free energy given in [2], [3] and [5] can be deduced from an axiom asserting that in a closed connection of a given history the material cannot do an arbitrarily large amount of useful work.

Acknowledgements. I gratefully acknowledge valuable discussions with Professor M. E. Gurtin.

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