THE IMPULSIVE STARTING OF A SPHERE*

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Abstract. A sphere is started impulsively in an incompressible viscous fluid of infinite extent. The Reynolds number is assumed to be large. The time of investigation is assumed to be small. The Navier–Stokes equations, written in spherical polar coordinates, are then iterated using the method of inner and outer expansions. Uniformly valid solutions are obtained to second order. The drag is also predicted.

1. Introduction. The boundary layer on an impulsively started sphere has been studied by Boltze [1] in his Göttingen thesis. Following Blasius [2], he iterated the planar boundary layer equations by assuming a balance between the unsteady acceleration forces and the viscous forces. However, Boltze's solution is valid only to the zeroth order. By applying the planar boundary layer equations to a curved surface, the higher order solutions obtained by him are incorrect in three respects: (i) The solutions are independent of Reynolds number due to the neglect of the curvature of the sphere. (ii) The slope of the inviscid 'outer' flow is not taken into consideration. This can be shown to have decisive effects on the shear stresses on the body. (iii) The effect of the boundary layer on the outer flow cannot be determined; consequently the pressure drag indicates zero. These effects are discussed by Wang [3].

The mutual interactions between the boundary layer and the inviscid 'outer' flow can be analyzed by the systematic application of the method of inner and outer expansions, as in the case of a circular cylinder [3]. Spherical polar coordinates will be used.

2. Solution. We can normalize the variables in the Navier-Stokes equations by the velocity of the sphere U_∞, the radius a, and the time $(\epsilon a/U_\infty)$. Here ϵ is a small number, since we concentrate our attention on the initial instant of starting where the sphere has moved only a fraction of the radius. We also assume large Reynolds numbers such that

$$R = U_\infty a/\nu = 1/\epsilon\alpha \gg 1, \tag{2.1}$$

where α is a constant of order unity. This method contains both a parameter expansion in Reynolds number and a coordinate expansion in time (Wang [4]). Then the Navier–Stokes equations become

$$r^2 \sin\theta\left(\frac{\partial \nabla^2 \psi}{\partial t} - \alpha\epsilon^2 \nabla^4 \psi\right) = \epsilon\left[2\nabla^2\psi \frac{\partial(\psi, \ln r \sin\theta)}{\partial(r, \theta)} - \frac{\partial(\psi, \nabla^2\psi)}{\partial(r, \theta)}\right], \tag{2.2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} - \cot\theta \frac{\partial}{\partial\theta}\right). \tag{2.3}$$

The initial conditions are

$$\psi(r, \theta, t) = 0, \qquad t \le 0, \tag{2.4}$$

*Received January 29, 1968. This paper was supported by the National Science Foundation, under Grant No. GP 6655.

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$$\psi(r,\,\theta,\,0^+) = \frac{1}{2}\sin^2\theta\left(r^2 - \frac{1}{r}\right). \tag{2.5}$$

On the body $r = 1$, the velocities are zero.

Equation (2.2) poses a singular perturbation problem. A direct expansion in the boundary layer thickness ϵ shows that the outer flow field is potential. Using the method of inner and outer expansions, a uniformly valid solution for the stream function is found:

$$\begin{aligned}\psi &= \Psi_0 + \epsilon\Psi_1 + \epsilon^2\Psi_2 + O(\epsilon^3)\\ &\quad + \epsilon(3\alpha^{1/2}t^{1/2}\sin^2\theta)[\pi^{-1/2}\exp(-\zeta^2) - \zeta\,\mathrm{erfc}\,\zeta]\\ &\quad + \epsilon^2(3\alpha t\sin^2\theta)[\zeta^2\,\mathrm{erfc}\,\zeta + \tfrac{1}{2}\,\mathrm{erfc}\,\zeta - \pi^{-1/2}\,\zeta\exp(-\zeta^2)]\\ &\quad + \epsilon^2(\tfrac{9}{2}\alpha^{1/2}t^{3/2}\sin^2\theta\cos\theta)g(\zeta) + O(\epsilon^3),\end{aligned} \tag{2.6}$$

where

$$\zeta = \tfrac{1}{2}(r-1)(\alpha t)^{-1/2}/\epsilon,$$

and

$$\begin{aligned}g(\zeta) &= 3\pi^{-1/2}\exp(-\zeta^2)\,\mathrm{erfc}\,\zeta - 2^{3/2}\pi^{-1/2}\,\mathrm{erfc}\,2^{1/2}\zeta - \zeta\,\mathrm{erfc}^2\,\zeta\\ &\quad - \pi^{-1/2}\left(\frac{7}{3} + \frac{4}{9\pi}\right)\exp(-\zeta^2) + \left(1 + \frac{2}{3\pi}\right)\zeta\,\mathrm{erfc}\,\zeta\\ &\quad + \frac{2}{3}\left(\frac{2}{3\pi} - 1\right)[\zeta^3\,\mathrm{erfc}\,\zeta - \pi^{-1/2}\zeta^2\exp(-\zeta^2)] + \frac{4}{3}\pi^{-1/2}\,\mathrm{erfc}\,\zeta.\end{aligned} \tag{2.7}$$

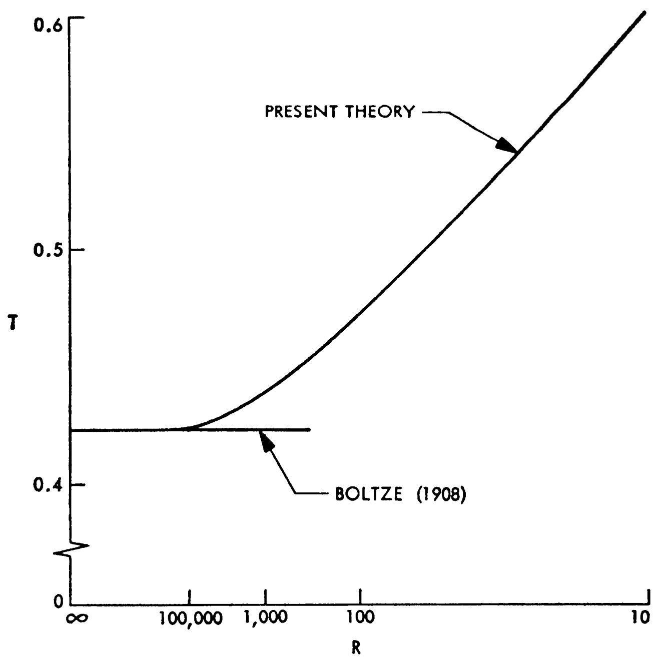

Fig. 1. The effect of Reynolds number on separation time.

Notice that only the last term in Eq. (2.6) corresponds to Boltze's inner solution. The function Ψ_0 can be identified as the uniform flow over a sphere, Eq. (2.5). Ψ_1 and Ψ_2 are induced outer flows, which are due to the displacement effects of the boundary layers:

$$\Psi_1 = -3\pi^{-1/2}(\alpha t)^{1/2} \sin^2\theta/r, \tag{2.8}$$

$$\Psi_2 = 9\pi^{-1/2}\left(2^{1/2} - 1 + \frac{2}{9\pi}\right)\alpha^{1/2}t^{3/2} \frac{\sin^2 \theta \cos \theta}{r^2} - \frac{3}{2}\alpha t \frac{\sin^2 \theta}{r}. \tag{2.9}$$

The terms which are proportional to $\sin \theta \cos \theta$ come from the nonlinear transport of vorticity. They give rise to the recirculating eddies or wake behind the sphere.

The separation time for each position θ is

$$T = \left\{\frac{-(\pi/R)^{1/2} - [\pi/R - 4(3 - 2/\pi) \cos \theta]^{1/2}}{2(3 - 2/\pi) \cos \theta}\right\}^2, \tag{2.10}$$

where T is the actual time normalized by (a/U_∞), see Fig. 1. Only in the limit of infinite

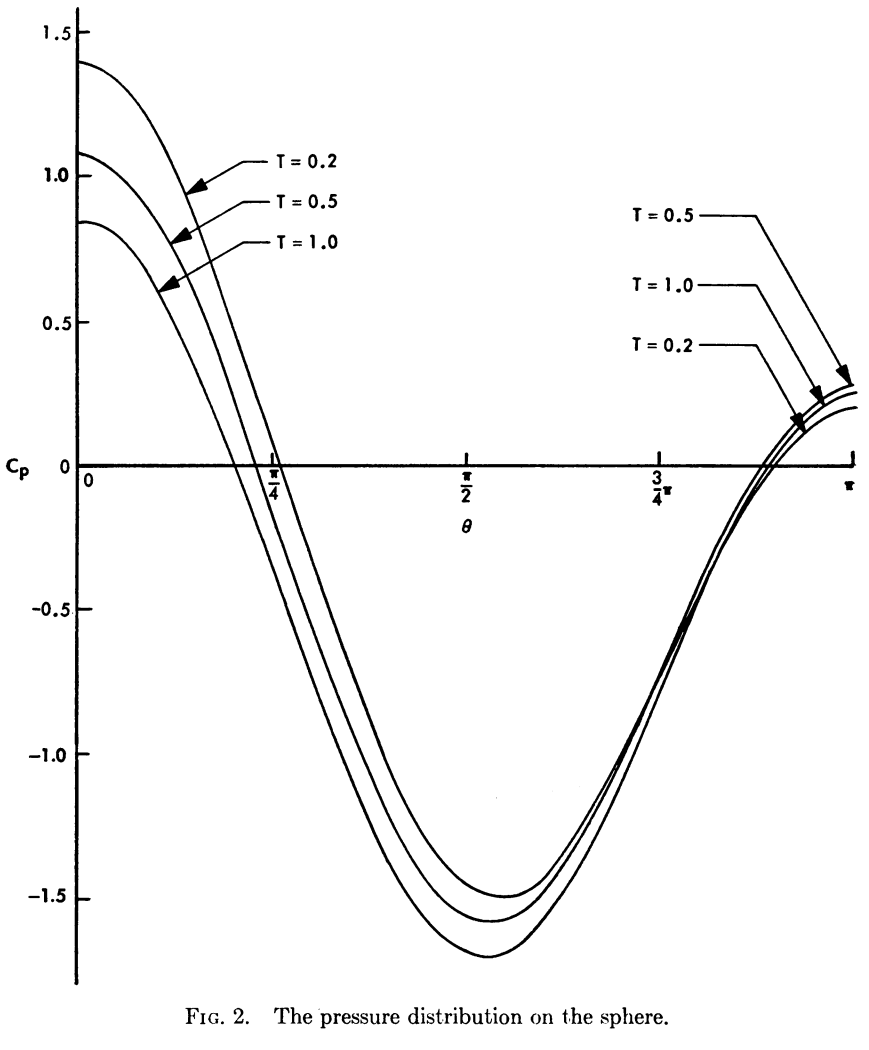

FIG. 2. The pressure distribution on the sphere.

Reynolds number does separation time reduce to Boltze's first order value $T = .423$. For finite Reynolds numbers the separation time (at $\theta = \pi$) is in general increased.

Taking the outer induced flow into consideration, the uniformly valid pressure distribution is found to be

$$\left(\frac{P - P_\infty}{\frac{1}{2}\rho U_\infty^2}\right) = 3\pi^{-1/2}(TR)^{-1/2}\frac{\cos\theta}{r^2} - 3\frac{\sin^2\theta}{r^3}\left(1 - \frac{1}{4r^3}\right) + \frac{1}{r^3}\left(2 - \frac{1}{r^3}\right)$$
$$+ \frac{3}{R}\frac{\cos\theta}{r^2} + \pi^{-1/2}\left(\frac{2}{\pi} + 9\cdot 2^{1/2} - 15\right)\left(\frac{T}{R}\right)^{1/2}\frac{(3\sin^2\theta - 2)}{r^3} + 3\pi^{-1/2}\left(\frac{T}{R}\right)^{1/2}\frac{(3\sin^2\theta - 4)}{r^6}$$
$$+ 9\left(\frac{T}{R}\right)^{1/2}\sin^2\theta\left[\zeta\,\mathrm{erfc}^2\,\zeta - 2\zeta\,\mathrm{erfc}\,\zeta + 2\pi^{-1/2}\exp(-\zeta^2)(1 - \mathrm{erfc}\,\zeta)\right.$$
$$\left. + \left(\frac{2}{\pi}\right)^{1/2}\mathrm{erfc}\,2^{1/2}\zeta\right] + O(\epsilon^2). \quad (2.11)$$

This is plotted in Fig 2.

It is found that the pressure drag is just half the shear drag. The total drag for impulsive start is

$$C_D = \frac{D}{\frac{1}{2}\rho U_\infty^2 \pi a^2} = 12\pi^{-1/2}(TR)^{-1/2} + 12R^{-1} + O(\epsilon^2). \quad (2.12)$$

This is plotted against time in Fig. 3, which also shows the experimental steady state values. However we must keep in mind that the solution is valid only for small times and cannot be extended to interpret the steady case. Notice that the time-independent term in Eq. (2.12) is the Stokes drag. This is because at small times the flow is essentially

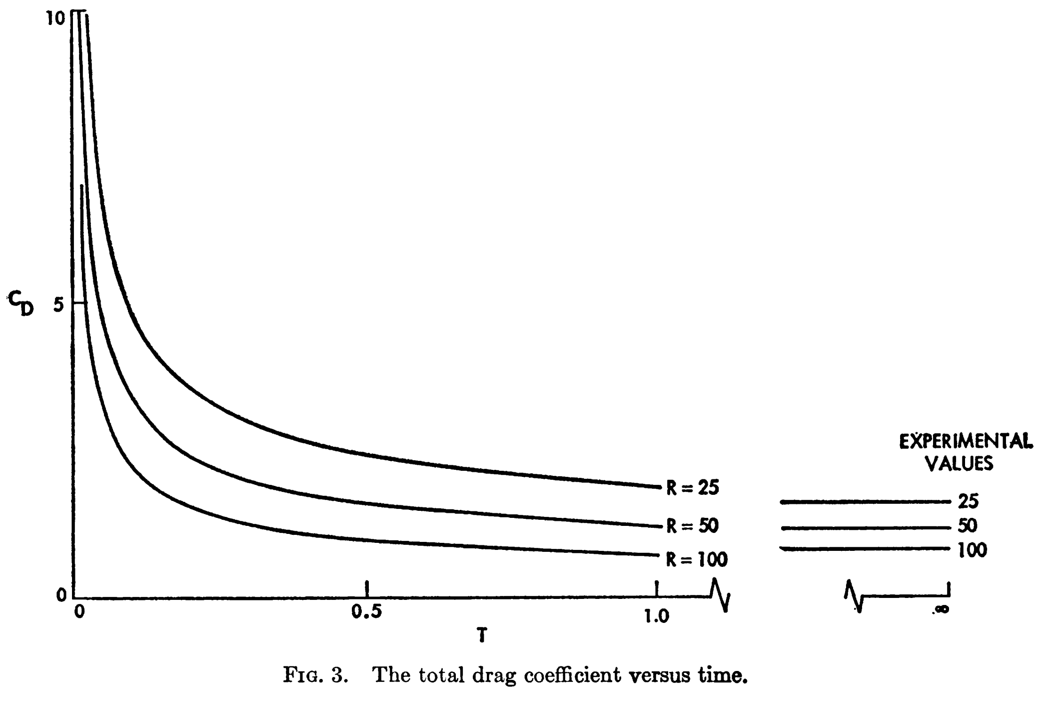

Fig. 3. The total drag coefficient versus time.

diffusive in nature. To the order considered the nonlinear convection affects only the stream function and the pressure, but not the drag.

The higher order corrections can be found, in principle, to any order. The process becomes increasingly tedious, since analytic solutions for each order must be obtained.

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