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ON THE TIME-DEPENDENT HEAT CONDUCTION AND THERMOELASTIC PROBLEMS*

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Introduction. An important problem in the theory of elasticity is the determination of stresses and displacements in a body subjected to thermal and mechanical loading. Complications arise in the instances of time-dependent heat generation, body forces, or boundary conditions. It has been customary to attack these problems using operational techniques [1], although in many cases inverse transforms are difficult to obtain, if indeed these inverses exist.

We present here a general solution to the uncoupled thermoelastic problem, i.e., a general solution to the heat conduction equation (including internal heat generation) to obtain the temperature distribution, and then the solution of the Navier equations to obtain the displacements of an elastic solid in the presence of the above temperature field and any mechanical loads. Time-dependent boundary conditions, body forces, and heat generation are easily included. The method, an extension of the Williams method [2], [3], makes use of the principle of superposition and classical mathematical techniques.

The heat conduction problem. The temperature distribution of an isotropic homogeneous solid with internal heat generation is governed by the well-known Fourier heat equation [4]:

$$k\nabla^2 T + Q/C = \partial T/\partial t. \tag{1}$$

A properly posed problem in heat conduction may be stated as follows: Find the temperature distribution $T(x, t)$ (note: we shall use x to denote the triplet of Cartesian coordinates x_1, x_2, x_3) in a body V, bounded by the surface S, satisfying the differential equation (1), and the following conditions:

$$\begin{aligned} &\text{in} \quad V, && T(x, 0) = T_0(x), \\ &\text{on} \quad S_1, && T(x, t) = f(x, t), \\ &\text{on} \quad S_2, && \frac{\partial T(x, t)}{\partial n} = g(x, t), \end{aligned} \tag{2}$$

where $S_1 + S_2 = S$.

For the homogeneous problem, where there is no internal heat generation and where $f = g = 0$ in Eqs. (2), it is known that temperature distributions may be sought in the form

$$T(x, t) = \sum_{n=1}^{\infty} \theta^{(n)}(x) e^{-\lambda_n t}. \tag{3}$$

The eigenfunctions $\theta^{(n)}(x)$ are the solutions to the differential equations

$$k\nabla^2 \theta^{(n)} + \lambda_n \theta^{(n)} = 0. \tag{4}$$

*Received July 1, 1967; revised version received February 2, 1968.

The eigenvalues λ_n are determined through the satisfaction of the homogeneous boundary conditions by the functions $\theta^{(n)}(x)$. In general there are a countable infinity of eigenvalues λ_n, each associated with a particular characteristic function. These shapes are orthogonal, i.e.,

$$\int_V \theta^{(m)}(x)\theta^{(n)}(x)\,dV = 0, \tag{5}$$

provided $\lambda_m \neq \lambda_n$, and also are of arbitrary amplitude. For the future development, we will remove this arbitrariness by imposing a normalization condition,

$$\int_V \theta^{(n)}(x)\theta^{(n)}(x)\,dV = 1. \tag{5a}$$

We shall now seek a solution to the full problem characterized by Eqs. (1) and (2). On the basis of the assumptions that the normalized eigenfunctions $\theta^{(n)}$ form a complete set, we shall construct a solution of the form

$$T(x, t) = T^{(s)}(x, t) + \sum_{n=1}^{\infty} \theta^{(n)}(x)q^{(n)}(t). \tag{6}$$

The function $T^{(s)}(x, t)$, termed the "static solution," satisfies

$$k\nabla^2 T^{(s)} + \frac{Q(x, t)}{C} = 0, \tag{7}$$

together with the boundary conditions (2). Thus by superposition, all the boundary conditions are satisfied by (6). The generalized time coordinate $q^{(n)}(t)$ introduced in (6) will be used to account for the initial condition (2) as well as the time variation of the boundary conditions and the heat generation through the static solution. Substituting the solution (6) into (1) and recalling (4) and (7) yield the result

$$\sum_{n=1}^{\infty} \theta^{(n)}(x)\{\dot{q}^{(n)}(t) + \lambda_n q^{(n)}(t)\} = -\frac{\partial T^{(s)}(x, t)}{\partial t}. \tag{8}$$

If we multiply (8) by the eigenfunction $\theta^{(n)}(x)$ and then make use of the orthonormality condition, we can reduce (8) to a first order differential equation for the generalized time coordinate:

$$\dot{q}^{(n)}(t) + \lambda n q^{(n)}(t) = \dot{Q}^{(n)}(t), \tag{9a}$$

where

$$Q^{(n)}(t) = -\int_V T^{(s)}(x, t)\theta^{(n)}(x)\,dV. \tag{9b}$$

At time $t = 0$, we note from (7) and (2) that

$$T_0(x) = T^{(s)}(x, 0) + \sum_{n=1}^{\infty} \theta^{(n)}(x)q^{(n)}(0). \tag{10}$$

Again making use of orthonormality conditions, we can determine the initial value of the generalized time coordinate to be

$$q^{(n)}(0) = Q^{(n)}(0) + \int_V T_0(x)\theta^{(n)}(x)\,dV. \tag{11}$$

In view of the modified initial condition (11), the solution to the ordinary differential equation (9a) is easily found to be

$$q^{(n)}(t) = \left[\int_V T_0(x)\theta^{(n)}(x)\, dV\right]e^{-\lambda_n t} + Q^{(n)}(t) - \lambda_n \int_0^t Q^{(n)}(\tau)e^{-\lambda_n(t-\tau)}\, d\tau. \qquad (12)$$

Thus the system of Eqs. (6), (7), (2) (4) and (12) forms the complete solution to the heat conduction problem posed by Eqs. (1) and (2). Thus we are able to specify a temperature distribution.

The thermoelastic problem. The motion of a heated elastic solid is governed by the Navier equations [5]

$$\mu U_{i,jj} + (\lambda + \mu)u_{j,ji} + F_i - (3\lambda + 2\mu)\alpha T, i = \rho\ddot{u}_i . \qquad (13)$$

Once the displacement field is known, the components of the stress tensor may be calculated from the relation

$$\tau_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) - (3\lambda + 2\mu)\alpha T\delta_{ij} . \qquad (14)$$

A well posed problem in dynamic thermoelasticity might be phrased as follows: Determine the displacement field in the volume V of an elastic solid bounded by the surface S satisfying equation (13), with the body force components F_i and a temperature distribution $T(x, t)$ specified in V, and the following additional conditions:

$$\begin{aligned} &\text{in} \quad V, && u_i(x, 0) = u_i^0(x) \quad \text{and} \quad \dot{u}_i(x, 0) = \dot{u}_i^0(x), \\ &\text{on} \quad S_1 , && u_i(x, t) = f_i(x, t), \\ &\text{on} \quad S_2 , && \overset{\nu}{T}_i = \tau_{ij}\nu_j = g_i(x, t), \end{aligned} \qquad (15)$$

where $S_1 + S_2 = S$, and ν_i are the components of the normal to S_2 , and $\overset{\nu}{T}_i$ is the stress vector.

In attempting a solution to the system of equations (13) and (15), we follow the approach outlined above and in [2], [3]. For the homogeneous problem, where the body forces vanish, the temperature distribution is isothermal, and the boundary conditions (15) are homogeneous, solutions to the equation (13) may be found in the form

$$u_i(x, t) = \sum_{n=1}^{\infty} U_i^{(n)}(x)e^{i\omega_n t}. \qquad (16)$$

The characteristic mode shapes are the solutions of the system of partial differential equations

$$\mu U_{i,jj}^{(n)} + (\lambda + \mu)U_{j,ji}^{(n)} + \rho\omega_n^2 U_i^{(n)} = 0. \qquad (17)$$

A denumerable set of eigenvalues are determined in the usual fashion. It is also well known that with each characteristic value ω_n there is associated a particular set of mode shapes $U_i^{(n)}(x)$. By standard methods we can determine the orthogonality condition, and if we adjoin again a normalization condition, we can write

$$\int_V \rho(x)U_i^{(n)}(x)U_i^{(m)}(x)\, dV = \delta_{nm} , \qquad (18)$$

where δ_{nm} is the Kronecker delta.

We shall now use the normalized eigenfunctions to find the solution to the full problem posed by (13) and (15). We assume that the displacement components can be expressed as

$$u_i(x, t) = u_i^{(s)}(x, t) + \sum_{n=1}^{\infty} U_i^{(n)}(x)q^{(n)}(t). \qquad (19)$$

The "static" displacements are the solutions of the differential equations

$$\mu u_{i,jj}^{(s)} + (\lambda + \mu)u_{j,ji}^{(s)} + F_i - (3\lambda + 2\mu)\alpha T_{,i} = 0 \tag{20}$$

that satisfy the complete boundary conditions (15). Again the principle of superposition indicates that the boundary conditions are completely satisfied by (19). To obtain the generalized time coordinate we note that the substitution of (19) into the full equation (13) yields:

$$\sum_{n=1}^{\infty} \rho(x)U_i^{(n)}(x)\{\ddot{q}^{(n)}(t) + \omega_n^2 q^{(n)}(t)\} = -\rho\ddot{u}_i^{(s)} . \tag{21}$$

As in the heat conduction problem, the equations (21) can be converted to relatively simple ordinary differential equations

$$\ddot{q}^{(n)} + \omega_n^2 q^{(n)} = \ddot{Q}^{(n)}(t), \tag{22a}$$

where

$$Q^{(n)}(t) = -\int_V \rho(x)u_i^{(s)}(x, t)U_i^{(n)}(x)\, dV. \tag{22b}$$

Making use of the initial conditions (15), together with standard techniques, we can write down the solution to the differential equation (22a) as

$$q^{(n)}(t) = \left[\int_V \rho u_i^0 U_i^{(n)}\, dV\right] \cos \omega_n t + \frac{1}{\omega_n}\left[\int_V \rho \dot{u}_i^0 U_i^{(n)}\, dV\right] \sin \omega_n t + Q^{(n)}(t) - \omega_n \int_0^t Q^{(n)}(\tau) \sin [\omega_n(t - \tau)]\, d\tau. \tag{23}$$

The system of Eqs. (19), (20), (15), (17) and (23) gives a complete solution to problems of dynamical thermoelasticity with time-dependent (as well as spatially-dependent) body forces, temperature distribution, and boundary conditions.

Conclusions. A method has been demonstrated whereby solutions for time-dependent problems in heat conduction and thermoelasticity may be obtained with only the use of classical linear mathematics. The key to this application of the principle of superposition is the separation of the original problem into a "static" problem, where the inhomogeneous terms are retained both in the boundary conditions and the differential equations while the inertia term is neglected, and a dynamic problem with homogeneous boundary conditions that is solved by an eigenfunction expansion that accounts for the initial conditions and the inertia effects of the static solution. Calculations to be presented elsewhere [3] indicate that the present expansion usually converges much more rapidly than conventional normal mode solutions.

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