A NECESSARY CONDITION FOR THE EXISTENCE OF SINGULARITY-FREE GLOBAL SOLUTIONS TO NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS*

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A dilatation invariance argument is employed to derive a necessary condition for the existence of singularity-free global solutions to nonlinear ordinary differential equations associated with a variational principle. It is shown that the necessary condition, in combination with the differential equation itself, is often sufficient to preclude the existence of such a solution.

Consider the generic class of second-order nonlinear ordinary differential equations of the Euler form

$$y'' \frac{\partial^2 F}{\partial y'^2} + y' \frac{\partial^2 F}{\partial y \, \partial y'} + \frac{\partial^2 F}{\partial x \, \partial y'} - \frac{\partial F}{\partial y} = 0, \tag{1}$$

where $F = F(x, y, y')$ is a prescribed C^2 real function with respect to its three real arguments and $y = y(x)$ is to be determined as a real singularity-free C^2 global solution of (1) either for all real x or for all real nonnegative x subject to prescribed boundary conditions at either $x = \pm\infty$ or at $x = 0, \infty$. Suppose that F admits a linear decomposition

$$\begin{gathered} F = \sum_{\{W\}} F^{(W)}(x, y, y'), \\ F^{(W)}(\lambda x, y, \lambda^{-1} y') \equiv \lambda^{W-1} F^{(W)}(x, y, y') \end{gathered} \tag{2}$$

for all real $\lambda > 0$, and that the functionals

$$J^{(W)}[y(x)] \equiv \int_D F^{(W)}(x, y, y') \, dx \tag{3}$$

exist for the solution $y(x)$ with D denoting the infinite or semi-infinite x domain; then we have $J^{(W)}[y(\lambda^{-1}x)] \equiv \lambda^W J^{(W)}[y(x)]$. Hence, as a consequence of the variational principle for (1)

$$\delta\Big(\sum_{\{W\}} J^{(W)}[y(x)]\Big) = 0 \quad \Rightarrow \quad \left(\frac{d}{d\lambda} \sum_{\{W\}} J^{(W)}[y(\lambda^{-1}x)]\right)_{\lambda=1} = 0, \tag{4}$$

it follows that the relation

$$\sum_{\{W\}} W J^{(W)}[y(x)] = 0 \tag{5}$$

is a necessary condition for the existence of such a singularity-free global solution $y(x)$. The following examples illustrate that (5) in combination with (1) is often sufficient to preclude the existence of such a solution and thus to obviate well-known but involved computational criteria for existence [1]–[3]. Moreover, the existence of singularity-free global solutions to certain higher-order nonlinear ordinary differential equations (asso-

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ciated with a variational principle) and to certain nonlinear partial differential equations (associated with a variational principle) can be precluded by similar analysis based on a relation of the form (5) [4].

FIRST EXAMPLE (embracing both Emden and Thomas-Fermi type equations):

$$x^{\alpha}y'' + \alpha x^{\alpha-1}y' + \gamma x^{\beta} y^{\gamma-1} = 0 \qquad [0 \leq x < \infty],$$

where $y(x)$ is nonnegative and α, β, γ are real constant parameters. We have $F = F^{(\alpha-1)} + F^{(\beta+1)}$ with $F^{(\alpha-1)} = x^{\alpha}(y')^2$ and $F^{(\beta+1)} = -2x^{\beta}y^{\gamma}$. For a solution such that the functionals (3) are finite, Eq. (5) produces

$$\int_0^{\infty} [(\alpha - 1)x^{\alpha}(y')^2 - 2(\beta + 1)x^{\beta}y^{\gamma}]\, dx = 0.$$

Integrating the first term in the integral by parts, using the differential equation, and assuming the boundary condition $\lim_{x\to 0} (x^{\alpha}yy') = 0$, we find

$$[(\alpha - 1)\gamma - 2(\beta + 1)] \int_0^{\infty} x^{\beta}y^{\gamma}\, dx = 0$$

which implies that the relation $(\alpha - 1)\gamma = 2(\beta + 1)$ is required for the existence of such a singularity-free global solution. Clearly, for γ an even integer this result holds with the nonnegativity of $y(x)$ relaxed. Special cases apply to the Emden equation ($\alpha = 2$, $\beta = 2, \gamma > 0$), the Thomas-Fermi equation ($\alpha = 0, \beta = -1/2, \gamma = 5/2$), the Bessel equation ($\alpha = 1, \beta = -1, \gamma = 2$), the Euler equation ($\alpha = 0, \beta = -2, \gamma = 2$), etc.

SECOND EXAMPLE (an equation of recent interest in elementary particle physics [5], [6]):

$$x^2y'' + 2xy' + x^2(y^{\gamma-1} - y) = 0 \qquad [0 \leq x < \infty].$$

We have $F = F^{(1)} + F^{(3)}$ with $F^{(1)} = x^2(y')^2$ and $F^{(3)} = x^2(y^2 - [2/\gamma]y^{\gamma})$. For a solution such that the functionals (3) are finite, Eq. (5) produces

$$\int_0^{\infty} x^2\left[(y')^2 + 3\left(y^2 - \frac{2}{\gamma}\, y^{\gamma}\right)\right] dx = 0.$$

Integrating the first term in the integral by parts and using the differential equation, we find

$$\int_0^{\infty} x^2\left[2y^2 + \left(1 - \frac{6}{\gamma}\right)y^{\gamma}\right] dx = 0,$$

which implies that no solution exists for γ an even integer greater or equal to 6 and no nonnegative solution exists for $\gamma \geq 6$.

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