TORSION OF THE SINGLE SPAN SUSPENSION BRIDGE[1]

BY

R. W. DICKEY

University of Wisconsin

1. Introduction. The purpose of this paper is to find the deflections of a single span suspension bridge under live loads (e.g. traffic) which are unsymmetrically distributed across the *width* of the suspension bridge stiffener truss (unsymmetric loading).

If the live load is distributed symmetrically across the *width* of the stiffener truss (symmetric loading) there exists a well-developed theory—the so-called deflection theory of suspension bridges (cf. [1, pp. 75ff.])—in which the support cables are approximated by a string and the stiffener truss is approximated by a beam of suitable flexural rigidity. Because of the nonlinearity of the deflection theory it is not in general possible to solve the deflection theory equation explicitly. However, solutions are given for certain special cases in the books by Steinman [2, pp. 264–267] and Johnson, Bryan, and Turneaure [3, pp. 290—293]. More recently Heller, Isaacson, and Stoker [4] have described a method for obtaining numerical solutions for bridges with arbitrary live loading and variable flexural rigidity and in the case of constant flexural rigidity Dickey [5] has proven the existence and uniqueness of solutions corresponding to a tension in the cable. However, the deflection theory is, by its nature, a two-dimensional theory and yields no information as to the behavior of a suspension bridge under unsymmetric loading. In view of this it would appear to be of interest to consider a theory in which unsymmetric loadings could be treated. For this purpose a generalization of the deflection theory will be developed in which the stiffener truss is treated as an indeterminate, pin-connected, three dimensional, elastic frame.

Sec. 2 of this paper will be devoted to describing the technique of finding the deflections (given the forces) of a three-dimensional linear elastic frame with many members. In Sec. 3 the cable equations will be combined with the truss equations and a method will be described, similar to that discussed in [4], for finding a solution of the resulting system of equations. Finally, in Sec. 3, a numerical example will be treated in which the deflections of a suspension bridge are found for both symmetric and unsymmetric live loads.

2. The stiffener truss. For the purposes of this paper a suspension bridge stiffener truss is assumed to be a periodic, three-dimensional, elastic frame, i.e. the truss is constructed of identical box-like modules, called bays, joined end to end along the truss axis.[2] The faces of each bay (cf. Fig. 1)—including those faces which are internal to the stiffener truss—have crossed diagonals. In reality stiffener trusses are built with statically determinate faces (although the whole truss is statically indeterminate); thus the example to be treated in this paper is more indeterminate than would normally be encountered in practice. To complete the description of the truss it will be assumed that

---

[1]Received November 16, 1967. Sponsored by the National Science Foundation, Contract No. GP-7814.

[2]In practice the stiffener truss is built in this manner although the strength of the members may vary from bay to bay.

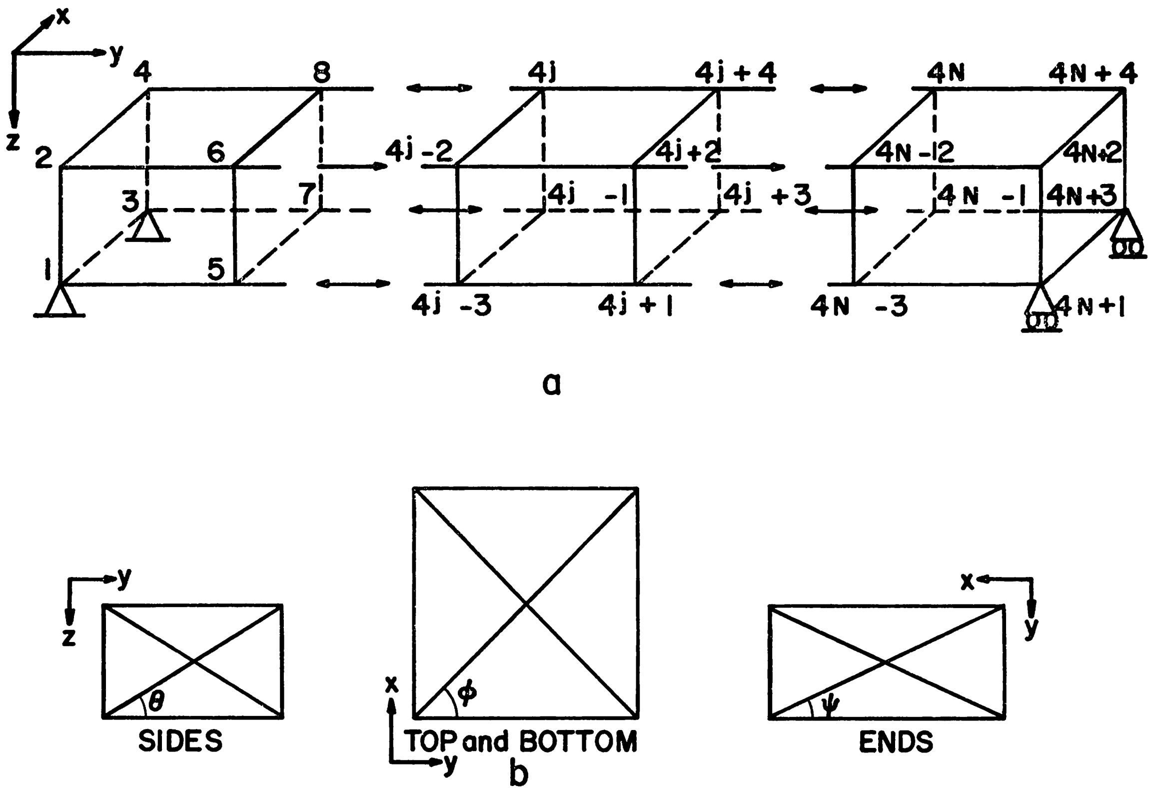

FIG. 1.

(cf. Fig. 1) the joints 1 and 3 are fixed, while the joints $4N + 1$ and $4N + 3$ (N bays) are free to roll in the axial direction. The object now is to discuss the reflections when vertical loads are applied the joints $4_j - 2, 4_j, 4_j + 2$, and $4_j + 4$, $j = 1, 2, \cdots, N$, i.e. when vertical loads are applied at the joints on top of the truss.

If vertical loads P_k are applied to the joints on top of the truss each joint, for example joint m, will be displaced an amount ξ_m, η_m, ζ_m, in the x, y and z directions. At the same time each bar in the truss will be deformed. Thus a bar of length $l_{i,n}$ connecting joints i and n will develop a strain $\epsilon_{i,n}$ and a stress $\sigma_{i,n}$. Assuming the joints are pin-connected (and that there is no local buckling) the total energy of the truss will be

$$W = \tfrac{1}{2} \sum \epsilon_{i,n}\sigma_{i,n}l_{i,n} + \sum P_k\zeta_k \tag{2.1}$$

where the first sum is taken over all bars in the truss and the second is taken over those joints on which there is an applied force. If the strains are small Hooke's Law may be assumed, i.e.

$$E_{i,n}\epsilon_{i,n} = \sigma_{i,n} \tag{2.2}$$

where $E_{i,n}$ is the Young's modulus for the bar (i, n). The strain in bar (i, n) will be defined as the ratio of the change in length to the original unstrained length, i.e.

$$\epsilon_{i,n} = \Delta l_{i,n}/l_{i,n} = \epsilon_{i,n}(\xi_i, \eta_i, \zeta_i, \xi_n, \eta_n, \zeta_n). \tag{2.3}$$

For example, to first order the strains in the bars $(4j - 3, 4j - 2)$ and $(4j - 2, 4j - 1)$ (cf. Fig. 1) are given by

$$\epsilon_{4j-3,\,4j-2} = (\zeta_{4j-3} - \zeta_{4j-2})/l_{12}$$

and

$$\epsilon_{4j-3,4j-2} = \frac{\sin \Psi(\zeta_{4j-1} - \zeta_{4j-2}) + \cos \Psi(\xi_{4j-1} - \xi_{4j-2})}{(l_{12}^2 + l_{13}^2)^{1/2}}.$$

From the preceeding remarks it is clear that (2.2) and (2.3) may be used to reformulate (2.1) in terms of the displacement. Thus

$$W = \tfrac{1}{2} \sum E_{i,n} l_{i,n} \epsilon_{i,n}^2(\xi_i\,, \eta_i\,, \zeta_i\,; \xi_n\,, \eta_n\,, \zeta_n) + \sum P_k \zeta_k\,. \tag{2.4}$$

The equation (2.4) formulates the energy of the truss as a (quadratic) function of the displacements. The equilibrium equations may be found from the requirement that

$$\partial W/\partial \xi_i = \partial W/\partial \eta_i = \partial W/\partial \zeta_i = 0, \quad (i = 2, 4, 5, 6, \cdots, 4N-1, 4N, 4N+2, 4N+4) \tag{2.5}$$

and

$$\partial W/\partial \eta_{4N+1} = \partial W/\partial \eta_{4N+3} = 0, \tag{2.6}$$

i.e. the derivatives are taken with respect to all the displacements except those which are fixed by the external constraints (for the truss in Fig. 1 those displacements which are fixed by the external constraints are ξ_1, η_1, ζ_1, ξ_3, η_3, ζ_3, ξ_{4N+1}, ζ_{4N+1}, ξ_{4N+1}, and ζ_{4N+3}). The resulting equations are linear in the displacements and may be written in the form

$$Su = f \tag{2.7}$$

where S is a $(12N + 2) \times (12N + 2)$ 'stiffness' matrix, u is the vector of displacements, and f is the vector of applied forces. It may be verified that the matrix S is symmetric and block-tridiagonal and in fact for the truss in Fig. 1 takes the form

$$\begin{bmatrix} A_1 & B_1 & & & & \\ B_1^T & A_2 & B_2 & & & 0 \\ & B_2^T & A_2 & B_2 & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & B_2^T & A_2 & B_3 \\ & & & & B_3^T & A_3 \end{bmatrix} \tag{2.8}$$

(the superscript T implies the transpose) where A_2, B_2, and B_2^T are 12×12, A_1 is 6×6, B_1 and B_1^T are 6×12 and 12×6, A_3 is 8×8, and B_3 and B_3^T are 12×8 and 8×12. The matrix (2.8) is written under the assumption that the corresponding members in each bay have the same elastic properties. However, if the properties vary from bay to bay the corresponding stiffness matrix will remain block-tridiagonal but will lose the periodic character which is exhibited in (2.8).

Various iteration schemes (cf. [4, pp. 64 ff]) were employed in an attempt to solve equation (2.7). However, it was found that while these schemes were successful for trusses having a few bays ($\approx$ 10), it was very difficult to attain convergence as the number of bays were increased. Since the object was to solve trusses with up to a hundred bays the iteration schemes were abandoned in favor of a more direct method. The method which was chosen makes use of the block-tridiagonal character of the coefficient matrix S to write (cf. [6, pp. 58 ff])

$$S = LU \tag{2.9}$$

where

$$L = \begin{bmatrix} \Delta_1 & & & & & 0 \\ B_1^T & \Delta_2 & & & & \\ & B_2^T & \Delta_3 & & & \\ & & \ddots & \ddots & & \\ 0 & & & B_2^T & \Delta_N & \\ & & & & B_3^T & \Delta_{N+1} \end{bmatrix} \tag{2.10}$$

and

$$U = \begin{bmatrix} I_1 & \Gamma_1 & & & 0 \\ & I_2 & \Gamma_2 & & \\ & & \ddots & \ddots & \\ & & & I_N & \Gamma_N \\ 0 & & & & I_{N+1} \end{bmatrix}. \tag{2.11}$$

I_i are identity matrices (I_1 is 6×6, I_2 through I_N are 12×12, and I_{N+1} is 8×8) and

$$\begin{aligned} &\text{(a)} && \Delta_1 = A_1, && \Gamma_1 = \Delta_1^{-1} B_1, \\ &\text{(b)} && \Delta_2 = A_2 - B_1^T \Gamma_1, && \Gamma_2 = \Delta_2^{-1} B_2, \\ &\text{(c)} && \Delta_i = A_2 - B_2^T \Gamma_{i-1}, && \Gamma_i = \Delta_i^{-1} B_2 \qquad (i = 3, \cdots, N-1), \\ &\text{(d)} && \Delta_N = A_2 - B_2^T \Gamma_{N-1}, && \Gamma_N = \Delta_N^{-1} B_3, \\ &\text{(e)} && \Delta_{N+1} = A_3 - B_3^T \Gamma_N . && \end{aligned} \tag{2.12}$$

It may be noted that because of the relatively small size of the matrices Δ_i (at most 12×12) it is feasible to calculate their inverses by Gauss elimination. Once those inverses have been calculated the matrices L and U may be found from (2.12) and in addition, since it is only necessary to invert the matrices Δ_i in order to invert L, the vector $L^{-1}f$ is easily calculated. Thus it only remains to solve $Uu = L^{-1}f$. However, U is upper triangular so that the solution of this system is also easily found.

The method which has been described above was actually carried out numerically on a CDC 6600 computer for trusses having a variety of dimensions and elastic properties. It was found that, even for trusses with a large number of bays (≈ 100), a complete solution of (2.7) could be found in a few seconds.

3. The suspension bridge. The object of this section is to couple the two cable equations to the equations for the stiffener truss which was discussed in Sec. 2 and to solve the resulting nonlinear system.

The vertical displacement of the cable $W_j(y)$ ($j = 1, 2$ distinguishes the two cables) due to live loads $r_j(x)$ on either side of the stiffener truss satisfy the differential equation (cf. [7, pp. 277 ff])

$$(H_j + h_j)\, d^2W_j/dy^2 = (h_j/H_j)q_j(y) - p_j(y) \quad (j = 1, 2) \tag{3.1}$$

with boundary conditions

$$W_j(0) = W_j(l) = 0, \quad (j = 1, 2) \tag{3.2}$$

where l is the length of the stiffener truss, $q_j(y)$ is that portion of the dead load (e.g. the weight of the stiffener truss and cables) which is supported by cable j, $p_j(y)$ is that portion of the live load which is supported by cable j, and H_j (constant) is the horizontal dead load tension in cable j. The term h_j, the induced live load tension, is given by (cf. [7, pp. 281 ff])

$$h_j = \frac{E_c A_c}{H_j l_c} \int_0^l W_j(y) q_j(y)\, dy \qquad (j = 1, 2) \tag{3.3}$$

where A_c and l_c are the cross-sectional area and length of the cables; E_c is the Young's modulus of the cables.[3]

The support cables of a suspension bridge are connected to the stiffener truss by short cables called hangers. However, since these hangers are placed at relatively short intervals it is reasonable to assume that the connection between the support cables and stiffener truss is continuous. In addition it will be assumed that the hangers are inextensible. Thus the vertical displacement of the stiffener truss at a point y will be the same as the vertical displacements of the cable at that point, e.g. (cf. Fig. 1)

$$W_1(nl_{15}) = \zeta_{4n+2} \qquad (n = 1, \cdots, N - 1) \tag{3.4a}$$

and

$$W_2(nl_{15}) = \zeta_{4n+4} \qquad (n = 1, \cdots, N - 1). \tag{3.4b}$$

The object now will be to replace the differential equations (3.1) by centered difference equations in which the interval size is taken as l_{15}, i.e. the length of a bay in the y direction. If this is done the equations (3.1) become

$$(H_1 + h_1)\{\zeta_{4n+6} - 2\zeta_{4n+2} + \zeta_{4n-2}\} + P_1(nl_{15})l_{15} = (h_1/H_1)q_1(nl_{15})l_{15} \quad (n = 1, \cdots, N - 1) \tag{3.5a}$$

and

$$(H_2 + h2)\{\zeta_{4n+8} - 2\zeta_{4n+4} + \zeta_{4n}) + P_2(nl_{15})l_{15} = (h_2/H_2)q_2(nl_{15})l_{15} \quad (n = 1, \cdots, N - 1) \tag{3.5b}$$

where from the boundary condition (3.2), $\zeta_2 = \zeta_4 = \zeta_{4N+2} = \zeta_{4N+4} = 0$.

Since the cables support a portion $P_j(y)$ of the live load it follows from the discussion in Sec. 2 that the deflections of the stiffener truss are a solution of

$$Su = r - p, \tag{3.6}$$

[3]For simplicity it has been assumed that the elastic properties of the two cables are the same.

where r is a vector containing $r_2(nl_{15})$ as elements and p is a vector containing $p_i(nl_{15})$ as elements. Thus the deflections of a suspension bridge are given by the solution of the equations (3.5), (3.6), and (3.3) for the unknown vector quantities u and p and the two scalar quantities h_1 and h_2 .

Before discussing the solution of the equations (3.5), (3.6) and (3.3) it is convenient to express the equations (3.5), and (3.6) as a single system. If this is done the resulting system takes the form

$$S'(h_1, h_2)u' = r' + q'(h_1, h_2), \tag{3.7}$$

where $S'(h_1, h_2)$ is a block-tridiagonal matrix, u' is a vector containing the elements of the vectors u and p, r' is a vector containing the elements of the vector r, and $q'(h_1, h_2)$ is a vector containing elements $h_1q_1(nl_{15})/H_1$, and $h_2q_2(nl_{15})/H_2$. Thus the problem of finding the deflections of a suspension bridge is reduced to solving the equations (3.3) and (3.7).

The nonlinear system of equations (3.3) and (3.7) were solved by iterating on the induced live load tensions h_1 and h_2 and solving (3.7) at each step by using the 'L-U' method described in Sec. 2. Thus at the nth step the vector u_n' was found by solving

$$S'(h_1^n, h_2^n)u_n' = r' - q'(h_1^n, h_2^n)$$

where $h_1^0 = h_2^0 = 0$. However, it was found that if at each step h_1^n and h_2^n were calculated from (3.3) (using the trapezoidal rule) and substituted directly into (3.8) the resulting iteration scheme converged to a solution having compressions in the cables. (A similar difficulty was encountered in [4] in attempting this type of iteration procedure on the deflection theory equations.) In order to find a solution having tensions in the cables a convergence parameter ω was introduced by defining a quantity γ_j^n as

$$\gamma_j^n = \frac{E_cA_c}{H_jl_c}\int_0^l q_j(y)W_j^n(y)\,dy \qquad (j = 1, 2) \tag{3.9}$$

and then determining $h_j^{(n+1)}$ from

$$h_j^{(n+1)} = (1 - \omega)h_j^n + \omega\gamma_j^n \qquad (j = 1, 2) \tag{3.10}$$

where as above $h_1^0 = h_2^0 = 0$. In the case of symmetric loading (identical loads on either side of the stiffener truss) it was possible to find a value of ω by trial and error such that the iteration scheme (3.8), (3.9), and (3.10) converged to a solution having tensions in the cables. However, if the loads are not the same on either side of the truss the simultaneous iterations on the induced live load tensions h_1 and h_2 no longer converge (in the case of symmetric loads $h_1 = h_2$, but this is not true if the loads are unsymmetric) and it is necessary to modify the procedure.

In order to attain convergence in the case of unsymmetric loads iterations on the induced live load tensions h_1 and h_2 were performed independently. Thus a value of h_2 , say $h_2 = h_2^i$, was determined and the following iteration scheme was employed:

$$S'(h_1^{i_n}, h_2^i)u_{i_n}' = r' - q'(h_1^{i_n}, h_2^i) \tag{3.11}$$

where $h_1^{i_{n+1}}$ is determined from

$$h_1^{i_{n+1}} = (1 - \omega_1)h_1^{i_n} + \omega_1\gamma_1^{i_n} \tag{3.12}$$

and

(3.13) $$\gamma_1^{i_n} = \frac{E_c A_c}{H_1 l_c} \int_0^l q_1(y) W_1^{i_n}(y)\, dy.$$

It was found that with the correct choice of ω_1 (ω_1 was found by trial and error) the above iteration scheme converged, i.e. $h_1^{i_n} \to h_1^i$ and $u'_{i_n} \to u'_i$. When convergence had been attained in (3.11), (3.12), and (3.13) a new value of h_2 , say h_2^{i+1} , was found from

(3.14) $$h_2^{i+1} = (1 - \omega_2)h_2^i + \omega_2\gamma_2^i$$

when

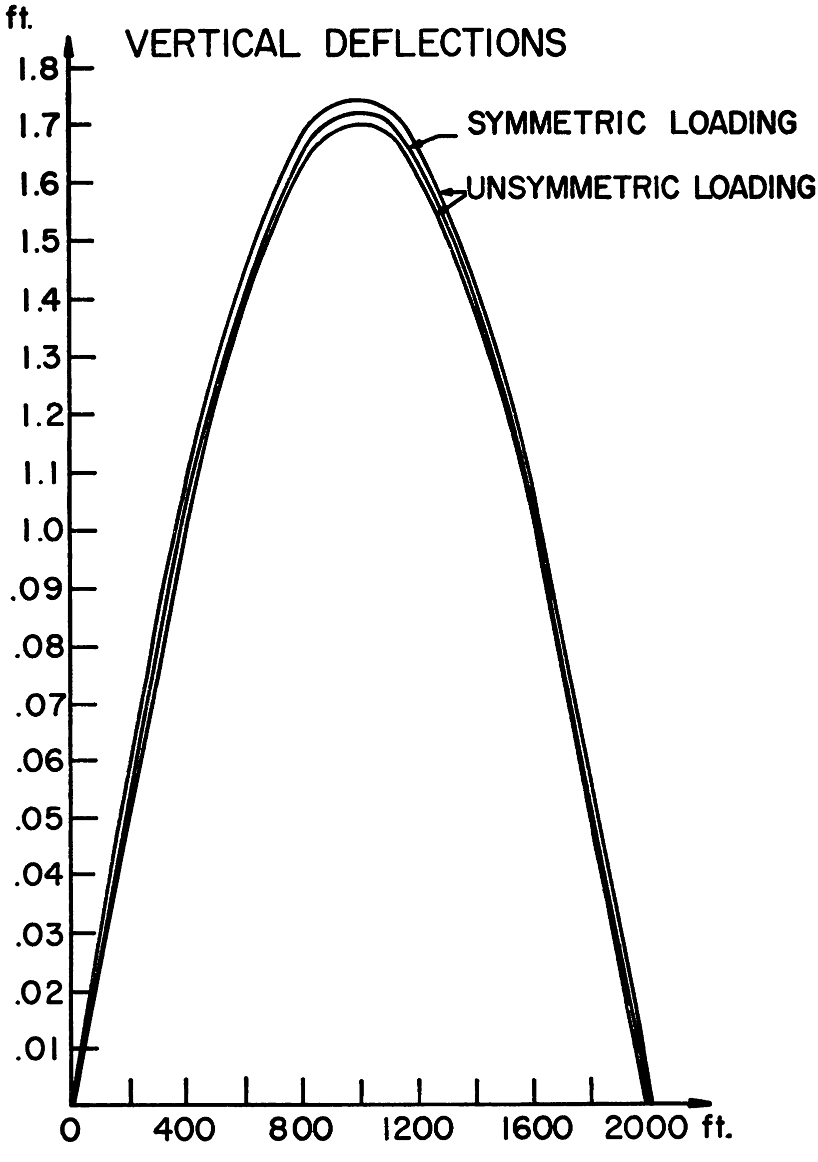

FIG. 2.

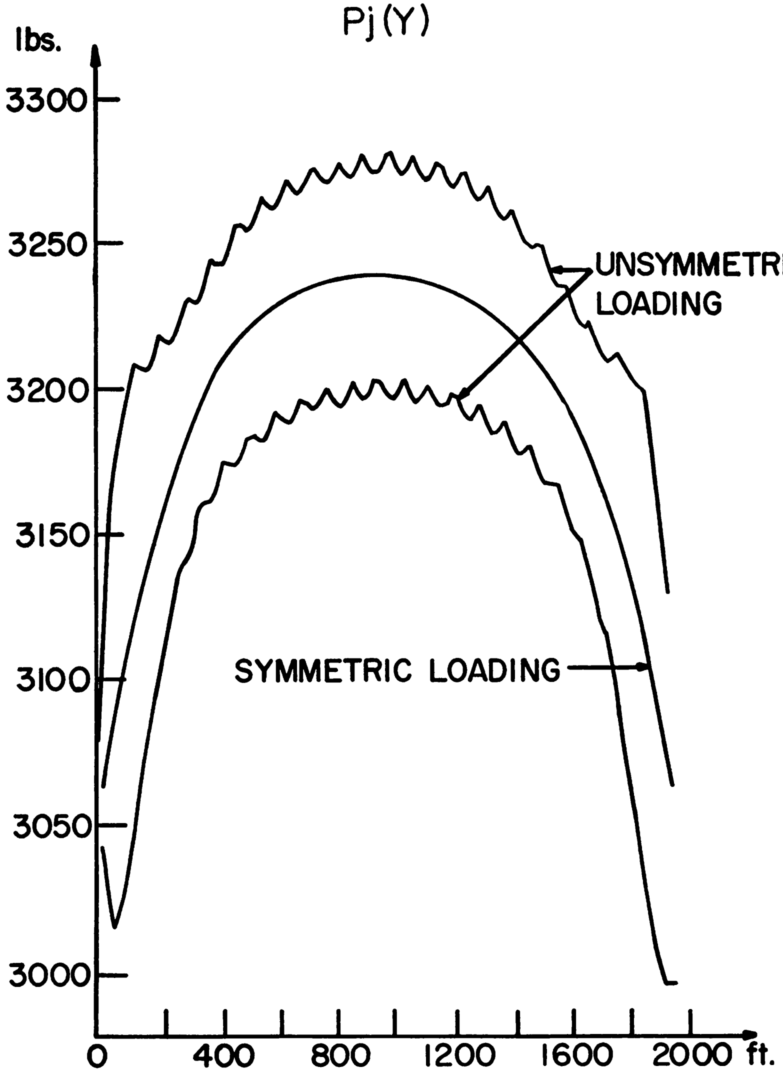

Fig. 3.

$$\gamma_2^i = \frac{E_c A_c}{H_2 l_c} \int_0^l q_2(y) W_2^i(y) \, dy. \tag{3.15}$$

The iteration scheme (3.11), (3.12), and (3.13) was then repeated with this new value of h_2 and $h_1^{i_\circ} = h_1^i$ (to begin the scheme $h_1^{0_\circ} = h_2^0 = 0$). It was found that it was possible to determine a value of ω_2 (by trial and error) such that the above iteration scheme converged for unsymmetrically loaded bridges of up to a hundred bays. In fact, even with a rough determination of ω_1 and ω_2 convergence of the above scheme for a hundred bays required only about ten minutes on a CDC 6600 computer.

In the preceding discussion a method has been described for finding the deflections of a suspension bridge under arbitrary live loads. It is now of interest to treat an example.

For this purpose a variety of problems were solved for a suspension bridge having a stiffener truss of length 2000 feet, depth 10 feet, width 100 feet, and constructed of one hundred bays. In Figs. 2 and 3 the vertical deflections and $p_j(y)$ (that portion of the live load which is supported by the cables) are plotted for both symmetric and unsymmetric loading. In both these cases the total live load on the stiffener truss was the same; however, in the symmetric case identical vertical live loads of 3250 lbs. were applied to every joint on the top of the truss, while in the unsymmetric case identical vertical live loads of 6500 lbs. were applied to the joints on one side of the truss and no load on the other side. In order to accentuate the torsional affects due to unsymmetric loading the strength of the internal members were reduced to a tenth of that of the other members of the stiffener truss. Even so it may be seen from Fig. 2 that under the moderate loadings which have been described the torsional affects are quite small. The oscillatory behavior of $p_j(y)$ (cf. Fig. 3) appears to be characteristic of unsymmetric loading. However, as the magnitude of the live load is increased the overall affect is to reduce the amplitude of these oscillations. Thus under sufficiently large live loads the oscillations become negligible and the curves of $p_j(y)$ versus y take on the overall appearance of parabolas.

During the construction of suspension bridges the situation may arise in which a portion of the stiffener truss is freely supported by the suspension cables (i.e. the ends of

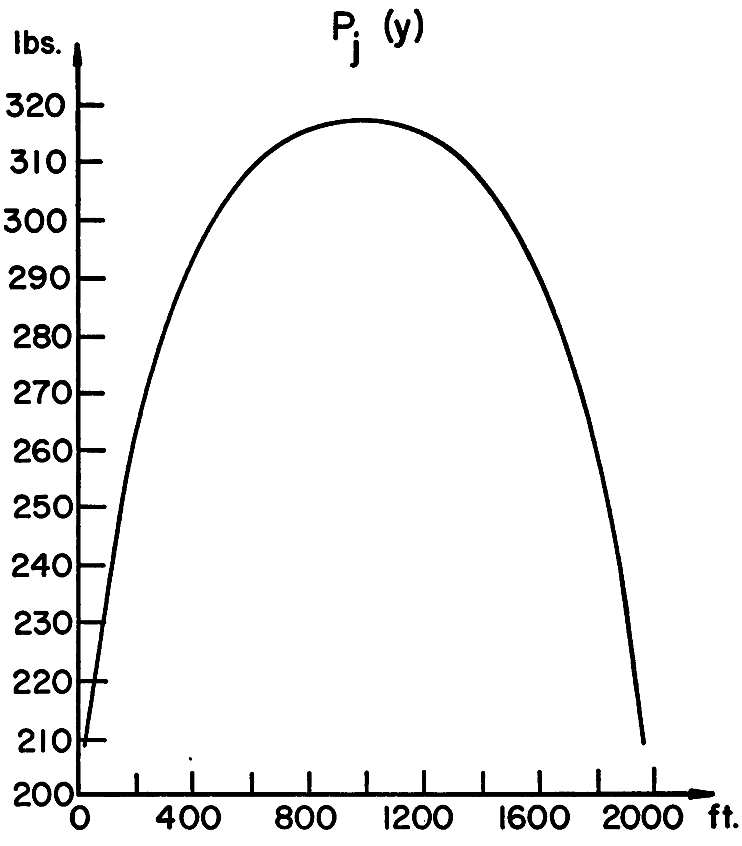

FIG. 4.

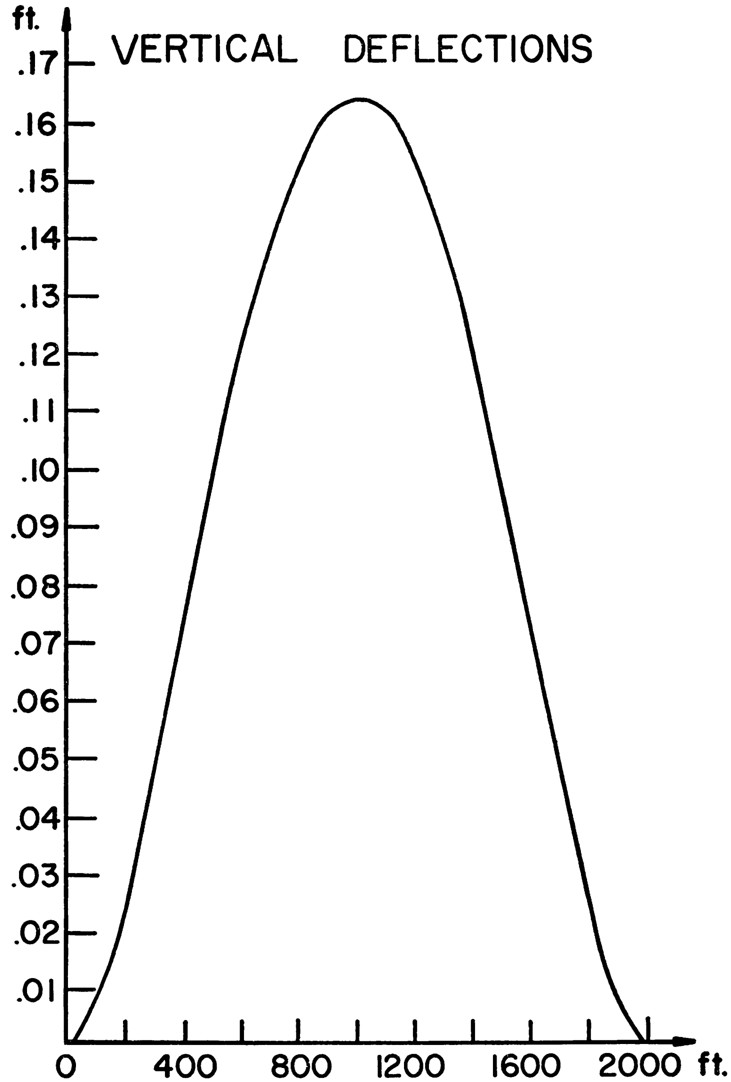

Fig. 5.

the stiffener truss are unsupported). The equations describing this modified system may be obtained by simply striking from the equations (3.8) those equations which refer to the members which have been removed, while retaining those equations which refer to the cable. (Note also that in computing $\gamma_1^{i''}$ and γ_2^i the dead loads $q_1(y)$ and $q_2(y)$ are zero over a portion of the interval of integration, cf. (3.13) and (3.15)). No difficulty was encountered in treating situations of this type by the method described above. As an example, the vertical deflections and $p_i(y)$ are plotted in Figs. 4 and 5 for a suspension bridge with unsupported ends. The physical characteristics of the truss and cables were identical to those which were used for the example in Figs. 2 and 3. The (symmetric) live load was 325 lbs. at each joint.

Acknowledgement. The author is indebted to E. Isaacson, H. B. Keller, and J. J. Stoker for their many important suggestions during the progress of this research.

References

[1] A. Pugsley, *The theory of suspension bridges*, Edward Arnold Ltd., London, 1957

[2] D. B. Steinman, *A practical treatise on suspension bridges*, 2nd Ed., Wiley, New York, 1929

[3] J. B. Johnson, C. W. Bryan and F. E. Turneaure, *The theory and practice of modern framed structures*, Part II, 9th Ed., Wiley, New York, 1911

[4] J. Heller, E. Isaacson and J. J. Stoker, *The numerical solution of the multispan suspension bridge deflection equations* (to appear)

[5] R. W. Dickey, *An improved numerical method for the solution of the suspension bridge deflection equations*, Math. of Comp. **102**, 298–303 (1968)

[6] E. Isaacson and H. B. Keller, *Analysis of numerical methods*, Wiley, New York, 1966

[7] T. Von Kármán and M. Biot, *Mathematical methods in engineering*, McGraw-Hill, New York, 1940