THE COMPLETENESS OF BIOT'S SOLUTION OF THE COUPLED THERMOELASTIC PROBLEM*

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Abstract. In one of his papers on the theory of thermoelasticity, M. A. Biot [2] has presented a solution to the differential equations very similar to the well-known Boussinesq–Papcovitch solution in the theory of elasticity. In this note it is proved that this solution is complete, the proof being based upon Mindlin's theorem of completeness of the Boussinesq–Papcovitch solution in elasticity.

1. Basic equations of thermoelasticity. In a three-dimensional space a bounded open region D is considered. A space point will be denoted by x and the boundary of D will be denoted by B. It is assumed that $B + D$ is a regular region (see Kellogg [1]). The mathematical formulation of the linear thermoelastic problem for an isotropic solid in the absence of body forces, inertia effects and internal heat supply consists of the following differential equations [2], holding for all $x \in D$, and for all values of the time variable larger than some initial value, which will be taken as zero:

$$(\lambda + \mu)\nabla\nabla\cdot\mathbf{u} + \mu\nabla^2\mathbf{u} - \beta\nabla\theta = 0, \qquad x \in D, \quad t > 0, \tag{1}$$

$$c_v\, \partial\theta/\partial t + \beta T_0\, \partial(\nabla\cdot\mathbf{u})/\partial t - k\nabla^2\theta = 0, \qquad x \in D, \quad t > 0, \tag{2}$$

where

- $\mathbf{u}$: displacement vector of the solid material,
- T_0: absolute reference temperature,
- θ: incremental temperature,
- t: time,
- λ, μ: Lamé's constants,
- $\beta = (\lambda + \frac{2}{3}\mu)\alpha$; α: coefficient of volumetric thermal expansion,
- c_v: specific heat per unit volume in the absence of deformations,
- k: thermal conductivity.

The vector function $\mathbf{u}$ and the scalar function θ will be assumed to be continuous, together with at least their first two partial space derivatives and their first partial time derivatives. It is furthermore assumed that the first partial time derivatives of these functions possess continuous partial space derivatives up to the second order at least. Functions having these properties will be said to be of class $C^{(2,1)}$. Functions of which only continuity in space and of the first n partial space derivatives is required will be said to be of class $C^{(n)}$. Clearly a function of class $C^{(n,m)}$ $(m \geq 0)$ is also of class $C^{(n)}$.

For a complete description of a particular problem a set of appropriate initial and boundary conditions must be given in addition to the differential equations. In this note, however, restriction is made to general solutions of the differential equations, regardless of the boundary and initial conditions.

*Received September 15, 1967.

As already noted by Biot, the theory of thermoelasticity is mathematically analogous to the theory of deformation of a fluid-saturated porous elastic material, which is often called the theory of consolidation of porous media. The quantity $\beta\theta$ in Eq. (1) is then replaced by the pressure of the fluid in the pores, and in the second equation the coefficients have a different physical meaning. The major difference between the two branches of science arises in applications, because in thermoelastic problems the first term in Eq. (2) usually dominates the second one, whereas in problems of consolidation the usual magnitude of the material constants results in the second term in Eq. (2) dominating the first. For a mathematical treatment this difference is of course irrelevant.

2. Biot's solution. A solution of the differential equations (1) and (2) has been given by Biot [2] in the following form:

$$\mathbf{u} = \nabla(\varphi + r\cdot\boldsymbol{\psi}) + \gamma\boldsymbol{\psi}, \tag{3}$$

$$\theta = [(\lambda + 2\mu)/\beta]\nabla^2\varphi + [2(\lambda + 2\mu)/\beta + \gamma(\lambda + \mu)/\beta]\nabla\cdot\boldsymbol{\psi}, \tag{4}$$

where γ is a constant related to the material properties by

$$\gamma = -2(\lambda + 2\mu + \beta^2 T_0/c_v)/(\lambda + \mu + \beta^2 T_0/c_v), \tag{5}$$

and where φ and $\boldsymbol{\psi}$ are a scalar and a vector function, respectively, which satisfy the following differential equations

$$\partial(\nabla^2\varphi)/\partial t = C\nabla^2\nabla^2\varphi, \tag{6}$$

$$\nabla^2\boldsymbol{\psi} = 0. \tag{7}$$

In Eq. (6) C represents another constant, which is related to the fundamental material properties by

$$C = \frac{k}{c_v}\frac{\lambda + 2\mu}{(\lambda + 2\mu + \beta^2 T_0/c_v)}. \tag{8}$$

Apart from some minor differences in notation the equations (3), (5), (6), (7), and (8) were given in Biot's paper [2]. The expression (4) for the incremental temperature was not given explicitly by Biot, but follows immediately from his considerations.

It can easily be verified by direct substitution from (3) and (4) into (1) and (2) that any pair of functions φ and $\boldsymbol{\psi}$ satisfying Eqs. (6) and (7) indeed represents a solution of the problem. The question naturally arises whether this solution is complete, or, in other words, whether any solution of the basic equations (1) and (2) admits a representation in the form of Eqs. (3) and (4). That this is indeed the case will be shown below.

3. Proof of completeness of Biot's solution. The starting point of the proof is the following theorem:

THEOREM 1. *For any solution* $\mathbf{v}_0$ *of the homogeneous linear problem*

$$(\lambda + \mu)\nabla\nabla\cdot\mathbf{v} + \mu\nabla^2\mathbf{v} = 0, \quad x \in D, \tag{9}$$

where D is the interior of a regular, bounded region and where $\mathbf{v}$ *is to be of class* $C^{(n)}$ *in* D $(n \geq 2)$, *there exists at least one pair of functions* f_0 , $\mathbf{F}_0$ *of class* $C^{(n)}$ *in* D, *such that*

$$\mathbf{v}_0 = \nabla(f_0 + \mathbf{r}\cdot\mathbf{F}_0) - 2[(\lambda + 2\mu)/(\lambda + \mu)]\mathbf{F}_0 , \qquad x \in D, \tag{10}$$

$$\nabla^2 f_0 = 0, \qquad x \in D, \tag{11}$$

$$\nabla^2 \mathbf{F}_0 = 0, \qquad x \in D, \tag{12}$$

where $\mathbf{r}$ *is the position vector of the point* x.

Equation (9) is the differential equation for the displacement vector in an isotropic linear elastic material in the absence of body forces. The solution (10) was given by Papcovitch [3], and independently, but somewhat later, by Neuber [4]. It is usually referred to, however, as the Boussinesq–Papcovitch solution because the fundamental parts of the solution were already presented by Boussinesq in his memoir on the theory of elasticity [5]. The presentation of the solution by Papcovitch already implied that it is a complete solution. An explicit proof of the completeness theorem stated above has been given by Mindlin [6] (see also Eubanks and Sternberg [7]). The proof is based upon Helmholtz' decomposition theorem which asserts that any vector function of class $C^{(n)}$ ($n \geq 0$) in a regular region can be represented as the sum of the divergence of a scalar function and the curl of a vector function, both functions being of class $C^{(n+1)}$ in that region (see Phillips [8]).

The pair of functions f_0, $\mathbf{F}_0$ may be replaced by a pair of functions h_0, $\boldsymbol{\psi}_0$ of the same class according to the following relations

$$h_0 = f_0 + [1 + (2/\gamma)(\lambda + 2\mu)/(\lambda + \mu)]\mathbf{r}\cdot\mathbf{F}_0, \qquad x \in D, \tag{13}$$

$$\boldsymbol{\psi}_0 = -(2/\gamma)(\lambda + 2\mu)/(\lambda + \mu)\mathbf{F}_0, \qquad x \in D, \tag{14}$$

where γ is an arbitrary nonzero constant. If it is furthermore assumed that γ is finite, then there exists a unique inverse relationship between h_0, $\boldsymbol{\psi}_0$ and f, $\mathbf{F}_0$,

$$f_0 = h_0 + [1 + (\gamma/2)(\lambda + \mu)/(\lambda + 2\mu)]\mathbf{r}\cdot\boldsymbol{\psi}_0, \qquad x \in D, \tag{15}$$

$$\mathbf{F}_0 = -(\gamma/2)(\lambda + \mu)/(\lambda + 2\mu)\boldsymbol{\psi}_0, \qquad x \in D. \tag{16}$$

Substitution of Eqs. (15) and (16) into Eqs. (10), (11) and (12) at once leads to the following generalization of Theorem 1.

Theorem 2. *For any solution* $\mathbf{v}_0$ *of the homogeneous linear problem*

$$(\lambda + \mu)\nabla\nabla\cdot\mathbf{v} + \nabla^2\mathbf{v} = 0, \quad x \in D,$$

where D *is the interior of a regular, bounded region and where* $\mathbf{v}$ *is to be of class* $C^{(n)}$ *in* D ($n \geq 2$), *and for any constant value of* γ *other than zero and infinite, there exists at least one pair of functions* h_0, $\boldsymbol{\psi}_0$ *of class* $C^{(n)}$ *in* D, *such that*

$$\mathbf{v}_0 = \nabla(h_0 + \mathbf{r}\cdot\boldsymbol{\psi}_0) + \gamma\boldsymbol{\psi}_0, \quad x \in D, \tag{17}$$

$$\nabla^2 h_0 + [2 + \gamma(\lambda + \mu)/(\lambda + 2\mu)]\nabla\cdot\boldsymbol{\psi}_0 = 0, \quad x \in D, \tag{18}$$

$$\nabla^2\boldsymbol{\psi}_0 = 0, \quad x \in D. \tag{19}$$

Equation (17) constitutes a somewhat more flexible representation than Eq. (10) because of the arbitrariness of γ. This is compensated, however, by the fact that the governing equation for h_0, Eq. (18), is more complicated than the one for f_0, Eq. (11).

Equation (1) is the inhomogeneous counterpart of Eq. (9), and therefore, in accordance with a fundamental theorem from the theory of linear differential equations (see e.g. Ford [9]), the complete solution of the inhomogeneous equation (1) consists of the sum of a particular solution of that equation and the complete solution of the corresponding homogeneous equation (9).

A particular solution $\mathbf{u}_*$ of the inhomogeneous equation

$$(\lambda + \mu)\nabla\nabla\cdot\mathbf{u} + \mu\nabla^2\mathbf{u} - \beta\nabla\theta_0 = 0, \quad x \in D, \tag{20}$$

where θ_0 is an arbitrary function of class $C^{(m)}$ in D ($m \geq 1$), and where $\mathbf{u}$ is to be of class $C^{(n)}$ in D ($n \geq 2$), will be sought in the form

$$\mathbf{u}_* = \nabla\varphi_*, \quad x \in D, \tag{21}$$

with φ_* a scalar function of class $C^{(n+1)}$. Substitution of (21) into (20) shows that this amounts to determining a function φ_* such that

$$\nabla[(\lambda + 2\mu)\nabla^2\varphi_* - \beta\theta_0] = 0, \quad x \in D,$$

which will surely be satisfied if the expression between brackets vanishes,

$$(\lambda + 2\mu)\nabla^2\varphi_* - \beta\theta_0 = 0, \quad x \in D. \tag{22}$$

This is Poisson's equation, a solution of which is (see e.g. Phillips [8])

$$\varphi_* = \frac{\beta}{\lambda + 2\mu}\int_D \frac{\theta_0\, dv}{r}, \qquad x \in D, \tag{23}$$

where r is the distance from the point x to the center of the volume element dv. In order that φ_* be of class $C^{(n+1)}$ it is sufficient to require that θ_0 is of class $C^{(n)}$ [8].

If now a function φ_0 is introduced as

$$\varphi_0 = h_0 + \varphi_*, \quad x \in D, \tag{24}$$

then this function is of class $C^{(n)}$ and it satisfies the equation

$$(\lambda + 2\mu)\nabla^2\varphi_0 + [2(\lambda + 2\mu) + \gamma(\lambda + \mu)]\nabla\cdot\boldsymbol{\psi}_0 - \beta\theta_0 = 0, \quad x \in D, \tag{25}$$

as follows immediately from (18), (22) and (24). Hence the following theorem has now been established.

Theorem 3. *For any solution $\mathbf{u}_0$ of the inhomogeneous linear problem*

$$(\lambda + \mu)\nabla\nabla\cdot\mathbf{u} + \mu\nabla^2\mathbf{u} - \beta\nabla\theta_0 = 0, \qquad x \in D,$$

where D is the interior of a regular, bounded region, $\mathbf{u}$ is to be of class $C^{(n)}$ in D ($n \geq 2$), and where θ_0 is an arbitrary function of class $C^{(n)}$ in D, and for any constant value of γ other than zero and infinite, there exists at least one pair of functions φ_0, $\boldsymbol{\psi}_0$ such that

$$\mathbf{u}_0 = \nabla(\varphi_0 + \mathbf{r}\cdot\boldsymbol{\psi}_0) + \gamma\boldsymbol{\psi}_0, \qquad x \in D, \tag{26}$$

$$(\lambda + 2\mu)\nabla^2\varphi_0 + [2(\lambda + 2\mu) + \gamma(\lambda + \mu)]\nabla\cdot\boldsymbol{\psi}_0 - \beta\theta_0 = 0, \qquad x \in D, \tag{27}$$

$$\nabla^2\boldsymbol{\psi}_0 = 0, \qquad x \in D. \tag{28}$$

So far Eq. (27) has been considered as an equation relating φ_0 to the functions $\boldsymbol{\psi}_0$ and θ_0, and the function θ_0 has been thought of as an arbitrary function, given in D. Alternatively, however, Eq. (27) may be considered as a representation for the arbitrary function θ_0 and in that case Theorem 3 asserts that for any pair of functions $\mathbf{u}_0$, θ_0 of which θ_0 is arbitrary and $\mathbf{u}_0$ is related to θ_0 through Eq. (20) there exists at least one pair of functions φ_0, ψ_0 of which ψ_0 is harmonic, but no restriction can be made concerning φ_0. The functions $\mathbf{u}_0$ and θ_0 can be derived from φ_0 and ψ_0 through Eqs. (26) and (27).

No use has yet been made of the fact that in order for the pair of functions $\mathbf{u}_0$, θ_0 to represent a solution of the coupled thermoelastic problem they have to satisfy not only Eq. (1) but also Eq. (2), i.e.

$$c_v\, \partial\theta_0/\partial t + \beta T_0\, \partial(\nabla\cdot\mathbf{u}_0)/\partial t - k\nabla^2\theta_0 = 0, \qquad x \in D, \quad t > 0. \tag{29}$$

This merely represents a restriction on the admissible functions θ_0, and therefore the representation (26), (27) is possible for any pair of functions $\mathbf{u}_0$, θ_0 satisfying Eqs. (1) and (2). Substitution of (26) and (27) into (29) shows that in order for the representation to be a solution one must have

$$\begin{aligned}(\lambda + 2\mu + \beta^2 T_0/c_v)\, \partial(\nabla^2\varphi_0)/\partial t &- (k/c_v)(\lambda + 2\mu)\nabla^2\nabla^2\varphi_0 \\ + [2(\lambda + 2\mu + \beta^2 T_0/c_v) + \gamma(\lambda + \mu + \beta^2 T_0/c_v)]\, \partial(\nabla\cdot\psi_0)/\partial t &= 0, \qquad x \in D,\ t > 0.\end{aligned} \tag{30}$$

In this equation the constant γ is still arbitrary. By choosing

$$\gamma = -\frac{2(\lambda + 2\mu + \beta^2 T_0/c_v)}{\lambda + \mu + \beta^2 T_0/c_v}, \tag{31}$$

the last term in Eq. (30) vanishes and one obtains

$$\partial(\nabla^2\varphi_0)/\partial t = C\nabla^2\nabla^2\varphi_0, \qquad x \in D, \quad t > 0, \tag{32}$$

where

$$C = \frac{k}{c_v}\,\frac{\lambda + 2\mu}{(\lambda + 2\mu + \beta^2 T_0/c_v)}. \tag{33}$$

Equation (32) constitutes the governing equation for φ_0. This function will be required to be of class $C^{(4,1)}$ in order to give meaning to the differential operations appearing in Eq. (32). From Theorem 3 with $n = 4$ and the considerations just given the following theorem is now obtained, restricting the admissible functions to those of class $C^{(4,1)}$, which form a subclass to the functions of class $C^{(4)}$.

THEOREM 4. *For any pair of functions* $\mathbf{u}_0$, θ_0, *being a solution of the differential equations* (1) *and* (2), *and both of class* $C^{(4,1)}$ *in D, with D the interior of a regular, bounded region, there exists at least one pair of functions* φ_0, ψ_0 *of class* $C^{(4,1)}$ *in D, satisfying the equations*

$$\partial(\nabla^2\varphi_0)/\partial t = C\nabla^2\nabla^2\varphi_0, \qquad x \in D, \quad t > 0,$$

$$\nabla^2\psi_0 = 0, \qquad x \in D, \quad t > 0,$$

where C is given by (33), *such that*

$$\mathbf{u}_0 = \nabla(\varphi_0 + \mathbf{r}\cdot\psi_0) + \gamma\psi_0, \qquad x \in D, \quad t > 0,$$

$$\theta_0 = [(\lambda + 2\mu)/\beta]\nabla^2\varphi_0 + [2(\lambda + 2\mu)/\beta + \gamma(\lambda + \mu)/\beta]\nabla\cdot\psi_0, \qquad x \in D, \quad t > 0,$$

where γ *is given by* (31).

Hence under the conditions of Theorem 4 Biot's solution is complete.

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